

BOUNDARY RELATIONS AND GENERALIZED RESOLVENTS OF SYMMETRIC OPERATORS

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ABSTRACT. The Kreĭn-Naimark formula provides a parametrization of all selfadjoint exit space extensions of a , not necessarily densely defined, symmetric operator, in terms of maximal dissipative (in \mathbb{C}_+) holomorphic linear relations on the parameter space (the so-called Nevanlinna families). The new notion of a boundary relation makes it possible to interpret these parameter families as Weyl families of boundary relations and to establish a simple coupling method to construct the generalized resolvents from the given parameter family. The general version of the coupling method is introduced and the role of boundary relations and their Weyl families for the Kreĭn-Naimark formula is investigated and explained.

1. INTRODUCTION

Let \mathfrak{H} be a separable Hilbert space and let A be a not necessarily densely defined closed symmetric operator or relation in \mathfrak{H} with equal defect numbers $n_+(A) = n_-(A) \leq \infty$. Denote by A^* the adjoint linear relation of A . The Kreĭn-Naimark formula

$$(1.1) \quad R_\lambda := P_{\mathfrak{H}}(\tilde{A} - \lambda)^{-1} \upharpoonright \mathfrak{H} = (A_0 - \lambda)^{-1} - \gamma(\lambda)(M(\lambda) + \tau(\lambda))^{-1}\gamma(\bar{\lambda})^*, \quad \lambda \in \rho(A_0) \cap \rho(\tilde{A}),$$

establishes a bijective correspondence between the set of all selfadjoint (canonical and exit space) extensions \tilde{A} of A and the set all Nevanlinna families $\tau(\lambda)$. Here $A_0 = A_0^*$ is a fixed canonical extension of A , $\gamma(\lambda)$ is the so-called γ -field, and $M(\lambda)$ is a Q -function of the pair $\{A, A_0\}$. The correspondence in (1.1) will also be indicated by the notation $\tilde{A} = A^{(\tau)}$. The Kreĭn-Naimark formula plays an important role in the extension theory of the operator A (see [1, 2, 3, 12, 17, 19, 24] and references therein) and its numerous applications to classical interpolation problems ([6, 31, 32, 33, 34, 39, 17, 19]), boundary value problems ([20, 21, 25]) and different type of physical problems (see [2, 3, 8, 9, 40, 42] and references therein).

During the last two decades a new approach to the extension theory, based on the concepts of boundary triplets and the corresponding Weyl functions, has been developed. Recall the basic definitions.

Definition 1.1. [25] A collection $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ consisting of a Hilbert space \mathcal{H} with $\dim \mathcal{H} = n_\pm(A)$ and two linear mappings Γ_0 and Γ_1 from A^* to \mathcal{H} , is said to be a boundary triplet for A^* if

(BT1) the abstract Green's identity holds

$$(1.2) \quad (f', g) - (f, g') = (\Gamma_1 \hat{f}, \Gamma_0 \hat{g})_{\mathcal{H}} - (\Gamma_0 f, \Gamma_1 g)_{\mathcal{H}}, \quad \hat{f} = \{f, f'\}, \quad \hat{g} = \{g, g'\} \in A^*;$$

(BT2) the linear mapping $\Gamma := \{\Gamma_0, \Gamma_1\} : A^* \rightarrow \mathcal{H} \oplus \mathcal{H}$ is surjective.

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The mappings Γ_0 and Γ_1 induce two selfadjoint extensions $A_0 = \ker \Gamma_0$ and $A_1 = \ker \Gamma_1$ of A . In [16, 17] the concept of a Weyl function was associated to an ordinary boundary triplet as an abstract version of the m -function appearing in boundary value problems for differential operators.

Definition 1.2. ([16, 17]) Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^* . The operator-valued function $M(\lambda)$ defined by

$$(1.3) \quad \Gamma_1 f_\lambda = M(\lambda) \Gamma_0 f_\lambda, \quad f_\lambda \in \mathfrak{N}_\lambda := \ker (A^* - \lambda), \quad \lambda \in \mathbb{C} \setminus \mathbb{R},$$

is said to be the *Weyl function*, corresponding to the triplet Π .

A connection between the approach via boundary triplets and the Kreĭn-Naimark theory of generalized resolvents has been established in [17, 19]. It was shown that all objects in (1.1) can be expressed in terms of boundary triplets. In fact, one has

$$(1.4) \quad A_0 = \ker \Gamma_0, \quad \gamma(\lambda) = (\Gamma_0 \lceil \mathfrak{N}_\lambda)^{-1}, \quad \{\Gamma_0, \Gamma_1\} R_\lambda f \in -\tau(\lambda), \quad f \in \mathfrak{H}.$$

In formula (1.1) the Weyl function $M(\lambda)$ is always a uniformly strict Nevanlinna function, whereas the parameter $\tau(\lambda)$ is an arbitrary Nevanlinna family. It is known that any uniformly strict Nevanlinna function is the Weyl function in the sense of Definitions 1.1 and 1.2.

If the parameter $\tau(\lambda)$ in formula (1.1) is a uniformly strict Nevanlinna function one can use the inverse problem for Weyl functions in order to construct the exit space extension $\tilde{A} = \tilde{A}^{(\tau)}$ connected with R_λ , via (1.1). This construction will be briefly recalled; cf. [12]. Let $S_1 := A$ and let S_2 be a symmetric operator in a Hilbert space \mathfrak{H}_2 such that $\tau(\lambda)$ is the Weyl function of S_2 corresponding to a boundary triplet $\Pi_2 = \{\mathcal{H}, \chi_0, \chi_1\}$. Then the linear relation

$$(1.5) \quad \tilde{A} = \left\{ \widehat{f}_1 \oplus \widehat{f}_2 \in A^* \oplus T_2 : \Gamma_0 \widehat{f}_1 - \chi_0 \widehat{f}_2 = \Gamma_1 \widehat{f}_1 + \chi_1 \widehat{f}_2 = 0 \right\}$$

is a selfadjoint (exit space) extension of $S_1 \oplus S_2$ and satisfies equation (1.1). Unfortunately this coupling approach was restricted to uniformly strict Nevanlinna functions $\tau(\lambda)$. In order to extend this method to arbitrary Nevanlinna families the new concepts of boundary relations and their Weyl families were introduced by the authors in [13], [15]. These concepts generalize the notions of the boundary triplet and the corresponding Weyl functions. In [13] it was proved that every Nevanlinna family $\tau(\lambda)$ can be realized as the Weyl family of a boundary relation. The main purpose of the paper is to show that, due to this new inverse result, the coupling construction in (1.5) can be extended to the case of any Nevanlinna family.

The paper is organized as follows. In Section 2 the basic notions are introduced and various preliminary results are established. In particular, some new and useful facts on unitary relations in Kreĭn spaces are presented, for instance, concerning the composition of unitary relations; see Theorem 2.13. In Section 3 the notion of boundary relations for S^* , the corresponding Weyl families, orthogonal couplings, and J -unitary transformations of boundary relations are discussed. In particular, it is shown that if two boundary relations $\tilde{\Gamma}$ and Γ are connected by means of a standard J -unitary operator W via $\tilde{\Gamma} = W\Gamma$, then the corresponding Weyl families are connected by means of Shmulyan's transform. Besides, the following equality is derived

$$(1.6) \quad \dim \mathcal{H} - n_\pm(A) = \text{mul } \Gamma,$$

showing, in particular, that the equality $\dim \mathcal{H} = n_{\pm}(A)$ is true if and only if Γ is single-valued.

In Section 4 the connection between boundary relation $(\tilde{\Gamma}, \mathcal{H}^2)$ and ordinary boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for S^* is investigated. In particular, it is shown (see Proposition 4.4) that formula

$$(1.7) \quad \tilde{\Gamma} = W\Gamma$$

establishes a bijective correspondence between the set of all boundary relations for S^* and the set of unitary relations W in $(\mathcal{H}^2, J_{\mathcal{H}})$ for which $\ker W = \{0\}$. Observe, that formula (1.7) leads to another (equivalent) definition of a boundary relation at the expense of extending the group of J -unitary operators in (1.7) to a (wider) set of J -unitary relations W with $\ker W = \{0\}$. In this section also generalized boundary triplets as well as boundary triplets whose Weyl functions take values in $[\mathcal{H}]$ are investigated.

In Section 5 there are some general transformation results concerning boundary relations $\Gamma : \mathfrak{H}^2 \rightarrow \mathcal{H}^2$ for S^* whose Weyl family $M(\lambda)$ belongs to the class $R[\mathcal{H}]$, that is $M(\cdot)$ is the Weyl function with values in $[\mathcal{H}]$. In this case an arbitrary orthogonal decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ of \mathcal{H} induces the corresponding block operator representation

$$(1.8) \quad M(\lambda) = (M_{ij}(\lambda))_{i,j=1}^2.$$

of $M(\cdot)$. It is shown how one can identify intermediate closed symmetric extensions H of A and associate boundary relations for H^* , such that the corresponding Weyl function is a given transform of the blocks of $(M_{ij}(\lambda))$ including, for instance, linear combinations of $M_{ij}(\lambda)$ and Schur complements. In particular, there appear induced boundary relations $\tilde{\Gamma}$ for H^* whose Weyl function $\tilde{M}(\cdot)$ equals either to $M_{11} + M_{22}$ or to $(M_{11} + M_{22})^{-1}$.

Similar results for ordinary boundary triplets $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for A^* have earlier been published in our previous paper [12]. However, the present generalizations are needed here for applications involving generalized resolvents.

In Section 6 the coupling method from [12], as briefly described above, is extended to the case of arbitrary Nevanlinna families $\tau(\cdot)$. This approach leads to new results and further geometric insight into various questions on this area. In the coupling method the selfadjoint exit space extension \tilde{A} in $\tilde{\mathfrak{H}} \supset \mathfrak{H}$ is constructed by means of a boundary triplet of A^* , whose Weyl function is $M(\cdot)$, and a boundary relation that corresponds to the family $\tau(\cdot) \in \tilde{R}(\mathcal{H})$. The coupling method makes it possible to treat the families $\tau(\cdot)$ and $-(\tau(\cdot) + M(\cdot))^{-1}$ appearing in (1.1) as the Weyl families of $S_2 := \tilde{A} \cap (\tilde{\mathfrak{H}} \ominus \mathfrak{H})^2$ (see formula (6.1)) and some intermediate extension of A (see formula (6.24)), respectively.

In Section 7 coupling method is applied to give a complete solution to the problem of M -admissibility (cf. [12]). Recall, that if A is nondensely defined, then $A^{(\tau)}$ may be either a (selfadjoint) linear relation or an operator (i.e, a single-valued linear relation). Based on a coupling construction, the following simple criterion for $\tau(\cdot)$ to generate an operator $A^{(\tau)}$ is established:

The Nevanlinna family $\tau(\cdot)$ in (1.1) corresponds to an operator $A^{(\tau)}$ (that is, it is Π -admissible) if and only if the following two conditions are satisfied:

$$(1.9) \quad w - \lim_{y \uparrow \infty} \frac{(\tau(iy) + M(iy))^{-1}}{y} = 0, \quad \lim_{y \uparrow \infty} \frac{(\tau(iy)^{-1} + M(iy)^{-1})^{-1}}{y} = 0.$$

Moreover, results on intermediate extensions given in Section 5 (a geometric treatment of $(\tau(iy) + M(iy))^{-1}$ as a Weyl function) allow us to show that if additionally A_0 (resp. A_1) is

an operator, then $A^{(\tau)}$ is an operator if and only if the first (resp. the second) of conditions (1.9) is satisfied.

In a forthcoming paper the coupling method is applied to the characterization of the Naimark extensions in terms of $\tau(\cdot)$.

2. PRELIMINARIES

2.1. Linear relations in Hilbert spaces. The *Cartesian product* $\mathfrak{H} \times \mathfrak{H}'$ of linear spaces \mathfrak{H} and \mathfrak{H}' is the set of all ordered pairs (of 1×2 matrices) $\{f, f'\}$ with $f \in \mathfrak{H}$ and $f' \in \mathfrak{H}'$. Frequently it will be convenient to denote the Cartesian product $\mathfrak{H} \times \mathfrak{H}'$ and the elements of it (as 2×1 matrices) by

$$\begin{pmatrix} f \\ f' \end{pmatrix} \in \begin{pmatrix} \mathfrak{H} \\ \mathfrak{H}' \end{pmatrix}, \quad f \in \mathfrak{H}, \quad f' \in \mathfrak{H}'.$$

If $\mathfrak{L} \subset \mathfrak{H}$ and $\mathfrak{L}' \subset \mathfrak{H}'$ are linear subspaces then $\mathfrak{L} \times \mathfrak{L}'$ denotes the Cartesian product of the subspaces; in agreement with the ordered pairs this product will also be denoted by $\{\mathfrak{L}, \mathfrak{L}'\}$, or by \mathfrak{L}^2 if $\mathfrak{L} = \mathfrak{L}'$.

A linear relation T from \mathfrak{H} to \mathfrak{H}' is a linear subspace of $\mathfrak{H} \times \mathfrak{H}'$. Systematically a linear operator T will be identified with its graph. It is convenient to write $T : \mathfrak{H} \rightarrow \mathfrak{H}'$ and interpret the linear relation T as a multi-valued linear mapping from \mathfrak{H} into \mathfrak{H}' . If $\mathfrak{H}' = \mathfrak{H}$ one speaks of a linear relation T in \mathfrak{H} .

For a linear relation $T : \mathfrak{H} \rightarrow \mathfrak{H}'$ the symbols $\text{dom } T$, $\ker T$, $\text{ran } T$, and $\text{mul } T$ stand for the domain, kernel, range, and the multi-valued part, respectively. The inverse T^{-1} is a relation from \mathfrak{H}' to \mathfrak{H} defined by $\{ \{f', f\} : \{f, f'\} \in T \}$. The adjoint T^* is the closed linear relation from \mathfrak{H}' to \mathfrak{H} defined by (see [5], [11])

$$(2.1) \quad T^* = \{ \{h, k\} \in \mathfrak{H}' \oplus \mathfrak{H} : (k, f)_{\mathfrak{H}} = (h, g)_{\mathfrak{H}'}, \{f, g\} \in T \}.$$

The sum $T_1 + T_2$ and the componentwise sum $T_1 \widehat{+} T_2$ of two linear relations T_1 and T_2 are defined by

$$\begin{aligned} T_1 + T_2 &= \{ \{f, g+h\} : \{f, g\} \in T_1, \{f, h\} \in T_2 \}, \\ T_1 \widehat{+} T_2 &= \{ \{f+h, g+k\} : \{f, g\} \in T_1, \{h, k\} \in T_2 \}. \end{aligned}$$

If the componentwise sum is orthogonal it will be denoted by $T_1 \oplus T_2$. The null spaces of $T - \lambda$, $\lambda \in \mathbb{C}$, are defined by

$$(2.2) \quad \mathfrak{N}_\lambda(T) = \ker(T - \lambda), \quad \widehat{\mathfrak{N}}_\lambda(T) = \{ \{f, \lambda f\} \in T : f \in \mathfrak{N}_\lambda(T) \}.$$

Moreover, $\rho(T)$ ($\widehat{\rho}(T)$) stands for the set of regular (regular type) points of T . The closure of a linear relation T will be denoted by $\text{clos } T$.

The product of linear relations is defined in the standard way. Some basic facts concerning the product of operators remain valid also for the product of relations. For instance, the following statement is easy to check.

Lemma 2.1. *Let \mathfrak{H}_1 , \mathfrak{H}_2 , and \mathfrak{H}_3 be Hilbert spaces and let $B : \mathfrak{H}_1 \rightarrow \mathfrak{H}_2$ and $A : \mathfrak{H}_2 \rightarrow \mathfrak{H}_3$ be linear relations, and let $C = AB$. Then:*

- (i) $\ker B \subset \ker C$ and $\text{mul } A \subset \text{mul } C$;
- (ii) if $\ker A = \{0\}$, then $\ker B = \ker C$, and if $\text{mul } B = \{0\}$, then $\text{mul } A = \text{mul } C$.

The next lemma gives some basic facts concerning the inverse, the product, and the adjoint of linear relations; these facts are well known for linear operators, the proofs for linear relations are left to the reader.

Lemma 2.2. *Let \mathfrak{H}_1 , \mathfrak{H}_2 , and \mathfrak{H}_3 be Hilbert spaces and let $B : \mathfrak{H}_1 \rightarrow \mathfrak{H}_2$ and $A : \mathfrak{H}_2 \rightarrow \mathfrak{H}_3$ be linear relations. Then:*

- (i) $(AB)^{-1} = B^{-1}A^{-1}$ and $(A^*)^{-1} = (A^{-1})^*$;
- (ii) $(AB)^* \supset B^*A^*$;
- (iii) if $A \in [\mathfrak{H}_1, \mathfrak{H}_2]$ or $B^{-1} \in [\mathfrak{H}_3, \mathfrak{H}_2]$, then $(AB)^* = B^*A^*$.

Recall that a linear relation T in \mathfrak{H} is called *symmetric (dissipative)* or *accumulative* if $\text{Im}(h', h) = 0$ (≥ 0) or ≤ 0 , respectively) for all $\{h, h'\} \in T$. These properties remain invariant under closures. By polarization it follows that a linear relation T in \mathfrak{H} is symmetric if and only if $T \subset T^*$. A linear relation T in \mathfrak{H} is called *selfadjoint* if $T = T^*$, and it is called *essentially selfadjoint* if $\text{clos } T = T^*$. A dissipative (accumulative) linear relation T in \mathfrak{H} is called maximal dissipative (maximal accumulative) if it has no proper dissipative (accumulative) extensions.

Assume that T is closed. If T is dissipative or accumulative, then $\text{mul } T \subset \text{mul } T^*$. In this case the orthogonal decomposition $\mathfrak{H} = (\text{mul } T)^\perp \oplus \text{mul } T$ induces an orthogonal decomposition of T as

$$(2.3) \quad T = T_s \oplus T_\infty, \quad T_\infty = \{0\} \times \text{mul } T, \quad T_s = \{\{f, g\} \in T : g \perp \text{mul } T\},$$

where T_∞ is a selfadjoint relation in $\text{mul } T$ and T_s is an operator in $\mathfrak{H} \ominus \text{mul } T$ with $\overline{\text{dom } T_s} = \overline{\text{dom } T} = (\text{mul } T^*)^\perp$, which is dissipative or accumulative. Moreover, if the relation T is maximal dissipative or accumulative, then $\text{mul } T = \text{mul } T^*$. In this case the orthogonal decomposition $(\text{dom } T)^\perp = \text{mul } T^*$ shows that T_s is a densely defined dissipative or accumulative operator in $(\text{mul } T)^\perp$, which is maximal (as an operator). In particular, if T is a selfadjoint relation, then there is such a decomposition where T_s is a selfadjoint operator (densely defined in $(\text{mul } T)^\perp$).

Let T be a linear relation in a Hilbert space \mathfrak{H} . If T is closed, then also the eigenspace $\widehat{\mathfrak{N}}_\lambda(T)$ is closed for every $\lambda \in \mathbb{C}$.

Lemma 2.3. [27] *Let T be a linear relation in \mathfrak{H} , let H be a restriction of T with a nonempty resolvent set, and assume that $\lambda \in \rho(H)$. Then H is closed and*

$$(2.4) \quad T = H \widehat{+} \widehat{\mathfrak{N}}_\lambda(T).$$

Observe that if S is a closed symmetric relation and H is a (maximal symmetric, maximal dissipative) selfadjoint extension, then

$$S^* = H \widehat{+} \widehat{\mathfrak{N}}_\lambda(S^*).$$

In particular, if S is a symmetric relation in \mathfrak{H} , then $H := S \widehat{+} \widehat{\mathfrak{N}}_\lambda(S^*)$ is a restriction of S^* and, moreover, if S is closed,

$$\bar{\lambda} \in \rho(H), \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

Hence, the adjoint relation S^* of a closed symmetric linear relation S in a Hilbert spaces \mathfrak{H} can be decomposed via the von Neumann formula:

$$(2.5) \quad S^* = S \widehat{+} \widehat{\mathfrak{N}}_\lambda(S^*) \widehat{+} \widehat{\mathfrak{N}}_{\bar{\lambda}}(S^*), \quad \lambda \in \mathbb{C} \setminus \mathbb{R}, \quad \text{direct sums},$$

where $\widehat{\mathfrak{N}}_\lambda(S^*)$ is defined as in (2.2). When $\lambda = \pm i$ the decomposition (2.5) is orthogonal:

$$(2.6) \quad S^* = S \oplus \widehat{\mathfrak{N}}_i(S^*) \oplus \widehat{\mathfrak{N}}_{-i}(S^*),$$

where the orthogonality is with respect to the inner product topology in S^* , cf. [5], [11]. A symmetric linear relation S is called *simple* if there is no nontrivial orthogonal decomposition of the Hilbert space $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$ and no corresponding orthogonal decomposition $S = S_1 \oplus S_2$ with S_1 a symmetric relation in \mathfrak{H}_1 and S_2 a selfadjoint relation in \mathfrak{H}_2 . The decomposition (2.3) for $S = S_s \oplus S_\infty$ shows that a simple closed symmetric relation is necessarily an operator. Recall that (cf. e.g. [37]) a closed symmetric linear relation S in a Hilbert space \mathfrak{H} is simple if and only if

$$\mathfrak{H} = \overline{\text{span}} \{ \mathfrak{N}_\lambda(S^*) : \lambda \in \mathbb{C} \setminus \mathbb{R} \}.$$

2.2. Linear relations in Kreĭn spaces. Recall that a signature operator j in a Hilbert space is a bounded linear operator such that $j = j^* = j^{-1}$. A signature operator provides the Hilbert space with a Kreĭn space structure with the fundamental symmetry j . Let \mathfrak{H} and \mathcal{H} be Hilbert spaces with signature operators $j_{\mathfrak{H}}$ and $j_{\mathcal{H}}$, respectively, and denote the corresponding Kreĭn spaces by $(\mathfrak{H}, j_{\mathfrak{H}})$ and $(\mathcal{H}, j_{\mathcal{H}})$. Then the adjoint $T^{[*]}$ of a linear relation T from the Kreĭn space $(\mathfrak{H}, j_{\mathfrak{H}})$ to the Kreĭn space $(\mathcal{H}, j_{\mathcal{H}})$ is given by $T^{[*]} = j_{\mathfrak{H}} T^* j_{\mathcal{H}}$.

The following result was given in [15, Proposition 2.2] for the Hilbert space case.

Proposition 2.4. *Let T be a closed linear relation from the Kreĭn space $(\mathfrak{H}, j_{\mathfrak{H}})$ to the Kreĭn space $(\mathcal{H}, j_{\mathcal{H}})$. Then:*

- (i) *$\text{dom } T$ is closed if and only if $\text{dom } T^{[*]}$ is closed;*
- (ii) *$\text{ran } T$ is closed if and only if $\text{ran } T^{[*]}$ is closed.*

Proof. The Kreĭn space adjoint $T^{[*]}$ of T is connected to the Hilbert space adjoint T^* via $T^{[*]} = j_{\mathfrak{H}} T^* j_{\mathcal{H}}$. Hence it is clear that $\text{dom } T^{[*]}$ ($\text{ran } T^{[*]}$) is closed if and only if $\text{dom } T^*$ (resp. $\text{ran } T^*$) is closed. Therefore, the statements follow from [15, Proposition 2.2]. \square

Definition 2.5. (i) A linear relation T from the Kreĭn space $(\mathfrak{H}, j_{\mathfrak{H}})$ to the Kreĭn space $(\mathcal{H}, j_{\mathcal{H}})$ is said to be *isometric* if $T^{-1} \subset T^{[*]}$ and *co-isometric* if $T^{[*]} \subset T^{-1}$.

(ii) A linear relation T is said to be *unitary* if it is simultaneously isometric and co-isometric, that is, if $T^{-1} = T^{[*]}$.

Lemma 2.6. ([15]) *Let T be a linear relation from the Kreĭn space $(\mathfrak{H}, j_{\mathfrak{H}})$ to the Kreĭn space $(\mathcal{H}, j_{\mathcal{H}})$. Then:*

- (i) *if $T : \mathfrak{K}_1 \rightarrow \mathfrak{K}_2$ isometric then the inverse $T^{-1} : \mathfrak{K}_2 \rightarrow \mathfrak{K}_1$ is isometric and the adjoint $T^{[*]} : \mathfrak{K}_2 \rightarrow \mathfrak{K}_1$ is co-isometric;*
- (ii) *if $T : \mathfrak{K}_1 \rightarrow \mathfrak{K}_2$ is unitary then the inverse T^{-1} and the adjoint $T^{[*]}$ are also unitary.*

Proof. (i) Since T is isometric, one has $T^{-1} \subset T^{[*]}$. Taking inverses one obtains

$$(T^{-1})^{-1} \subset (T^{[*]})^{-1} = (T^{-1})^{[*]}$$

by Lemma 2.2, so that T^{-1} is isometric. Taking adjoints one obtains

$$(T^{[*]})^{[*]} \subset (T^{-1})^{[*]} = (T^{[*]})^{-1},$$

again by Lemma 2.2, so that $T^{[*]}$ is co-isometric.

(ii) This statement is clear from $(T^{-1})^{[*]} = (T^{[*]})^{-1}$; cf. Lemma 2.2. \square

The following two statements are due to Yu.L. Shmul'jan [43]. They can be obtained also directly from the equality $T^{[*]} = T^{-1}$ and Proposition 2.4; see also [15].

Proposition 2.7. *Let T be a unitary relation from the Kreĭn space $(\mathfrak{H}, j_{\mathfrak{H}})$ to the Kreĭn space $(\mathcal{H}, j_{\mathcal{H}})$. Then:*

- (i) $\text{dom } T$ is closed if and only if $\text{ran } T$ is closed;
- (ii) the following equalities hold:

$$(2.7) \quad \ker T = (\text{dom } T)^{[\perp]}, \quad \text{mul } T = (\text{ran } T)^{[\perp]}.$$

A unitary relation $T : (\mathfrak{H}, j_{\mathfrak{H}}) \rightarrow (\mathcal{H}, j_{\mathcal{H}})$ may be multi-valued, nondensely defined, or unbounded. The following characterization is useful.

Lemma 2.8. ([15]) *Let T be a unitary relation from the Kreĭn space $(\mathfrak{H}, j_{\mathfrak{H}})$ to the Kreĭn space $(\mathcal{H}, j_{\mathcal{H}})$. Then:*

- (i) T is single-valued if and only if $\overline{\text{ran}} T = \mathcal{H}$;
- (ii) T is single-valued and densely defined if and only if $\overline{\text{ran}} T = \mathcal{H}$ and $\ker T = \{0\}$;
- (iii) T is single-valued and bounded (not necessarily densely defined) if and only if $\text{ran } T = \mathcal{H}$;
- (iv) $T \in [\mathfrak{H}, \mathcal{H}]$ if and only if $\text{ran } T = \mathcal{H}$ and $\ker T = \{0\}$.

A unitary relation T is the graph of an operator if and only if its range is dense. In this case it need not be densely defined or bounded; and if it is bounded it need not be densely defined.

Corollary 2.9. *Let T be a unitary relation from the Kreĭn space $(\mathfrak{H}, j_{\mathfrak{H}})$ to the Kreĭn space $(\mathcal{H}, j_{\mathcal{H}})$. Then $T \in [\mathfrak{H}, \mathcal{H}]$ if and only if $T^{-1} \in [\mathcal{H}, \mathfrak{H}]$.*

Proof. Assume $T \in [\mathfrak{H}, \mathcal{H}]$. Then $\text{dom } T = \mathfrak{H}$ and $\text{mul } T = \{0\}$, or equivalently, $\text{ran } T^{-1} = \mathfrak{H}$ and $\ker T^{-1} = \{0\}$. By (iv) of Lemma 2.8 this implies that $T^{-1} \in [\mathcal{H}, \mathfrak{H}]$. \square

Observe that for a unitary relation T from $(\mathfrak{H}, j_{\mathfrak{H}})$ to $(\mathcal{H}, j_{\mathcal{H}})$, both T and T^{-1} are operators if and only if $\overline{\text{dom}} T = \mathfrak{H}$ and $\overline{\text{ran}} T = \mathcal{H}$. Moreover, in this case $\text{dom } T = \mathfrak{H}$ if and only if $\text{ran } T = \mathcal{H}$, cf. Proposition 2.7, which also leads to Corollary 2.9.

Remark 2.10. In the present terminology an operator T is unitary if it satisfies $T^{-1} = T^{[*]}$. However, the terminology in [4, Chapter 2, Definition 5.1 and Corollary 5.8] is different. An operator T from the Kreĭn space $(\mathfrak{H}, j_{\mathfrak{H}})$ to the Kreĭn space $(\mathcal{H}, j_{\mathcal{H}})$ is unitary in the sense of M.G. Kreĭn (see[4]), if $\text{dom } T = \mathfrak{H}$, $\text{ran } T = \mathcal{H}$, and

$$(2.8) \quad [Tf, Tf]_{\mathcal{H}} = [f, f]_{\mathfrak{H}}, \quad f \in \mathfrak{H}.$$

To see the connection with the present setting, observe that (2.8) implies by polarization that

$$(2.9) \quad [Tf, Tg]_{\mathcal{H}} = [f, g]_{\mathfrak{H}}, \quad f, g \in \mathfrak{H}.$$

The identity (2.9) shows that T is isometric, i.e., the graph of T satisfies $T^{-1} \subset T^{[*]}$, and that $\ker T = \{0\}$. Since $\text{dom } T = \mathfrak{H}$ and $\text{ran } T = \mathcal{H}$ it follows that T is unitary, i.e., $T^{-1} = T^{[*]}$, cf. [15, Proposition 2.5]. Moreover, $T \in [\mathfrak{H}, \mathcal{H}]$ by (iv) of Lemma 2.8 and then also $T^{-1} \in [\mathcal{H}, \mathfrak{H}]$ by Corollary 2.9. Conversely, if T is a unitary relation, i.e., $T^{-1} = T^{[*]}$ and $T \in [\mathfrak{H}, \mathcal{H}]$ (or equivalently $T^{-1} \in [\mathcal{H}, \mathfrak{H}]$), then T is an operator satisfying (2.8), $\text{dom } T = \mathfrak{H}$, and $\text{ran } T = \mathcal{H}$ by (iv) of Lemma 2.8. Therefore, a unitary relation T is a *standard unitary operator* (in the sense of M.G. Kreĭn) precisely when T in addition belongs to $[\mathfrak{H}, \mathcal{H}]$, i.e.,

T is everywhere defined and single-valued, in which case also $T^{-1} \in [\mathcal{H}, \mathfrak{H}]$ is a standard unitary operator. In the present paper a unitary operator need not belong to $[\mathfrak{H}, \mathcal{H}]$ and it need not be even densely defined, in which case $\ker T$ is also nontrivial; cf. Proposition 2.7.

On a finite-dimensional space the set of injective unitary operators coincides with the set of standard unitary operators.

Corollary 2.11. *Let \mathfrak{K}_1 and \mathfrak{K}_2 be Kreĭn spaces, let $T : \mathfrak{K}_1 \rightarrow \mathfrak{K}_2$ be a unitary operator with $\ker T = \{0\}$, and assume that $\dim \mathfrak{K}_1 < \infty$. Then $\dim \mathfrak{K}_2 = \dim \mathfrak{K}_1 < \infty$ and T is a standard unitary operator in $[\mathfrak{K}_1, \mathfrak{K}_2]$.*

Proof. Since $\ker T = \{0\}$ and $\dim \mathfrak{K}_1 < \infty$, it follows from Proposition 2.7 that $\text{dom } T = \mathfrak{K}_1$ and that $\text{ran } T$ is closed; furthermore, since T is single-valued, one has $\text{ran } T = \mathfrak{K}_2$. Thus $T \in [\mathfrak{K}_1, \mathfrak{K}_2]$ by (iv) of Lemma 2.8, so that T is a standard unitary operator; cf. Remark 2.10. Therefore also $\dim \mathfrak{K}_2 = \dim \mathfrak{K}_1 < \infty$. \square

It is emphasized that the condition $\ker T = \{0\}$ in Corollary 2.11 is essential: as will be seen below (see also [15]) boundary triplets of symmetric operators S , $\text{dom } S \neq \{0\}$, (acting on a finite-dimensional or infinite dimensional space) are typical examples of bounded unitary operators which are not standard. They are nondensely defined with a nontrivial kernel that is equal to S .

Unitary relations between Kreĭn spaces admit a couple of useful properties under composition. First a result which concerns the adjoint of the product of linear relations in the case that the domain or the range of one of the relations is closed; observe that Lemma 2.2 is still true in the Kreĭn space situation.

Lemma 2.12. *Let \mathfrak{K}_j , $j = 0, 1, 2, 3$, be Kreĭn spaces and let $S : \mathfrak{K}_1 \rightarrow \mathfrak{K}_2$ be a closed relation. Then:*

(i) *if $\text{dom } S$ is closed then for every linear relation $X : \mathfrak{K}_0 \rightarrow \mathfrak{K}_1$ with $\text{ran } X \subset \text{dom } S$ one has*

$$(SX)^{[*]} = X^{[*]}S^{[*]};$$

(ii) *if $\text{ran } S$ is closed then for every linear relation $Y : \mathfrak{K}_2 \rightarrow \mathfrak{K}_3$ with $\text{dom } Y \subset \text{ran } S$ one has*

$$(YS)^{[*]} = S^{[*]}Y^{[*]}.$$

Proof. (i) The inclusion $(SX)^{[*]} \supset X^{[*]}S^{[*]}$ is always satisfied, cf. (ii) in Lemma 2.2. To prove the reverse inclusion let $\{f, g\} \in (SX)^{[*]}$, so that

$$(2.10) \quad [g, h]_{\mathfrak{K}_1} = [f, k]_{\mathfrak{K}_3} \quad \text{for all } \{h, k\} \in SX.$$

Since the linear relation SX contains the set

$$\{\{0, f_0\} : f_0 \in \text{mul } S\}$$

it follows from (2.10) that $[f, f_0] = 0$ for all $f_0 \in \text{mul } S$, so that $f \in (\text{mul } S)^{\perp} = \overline{\text{dom } S^{[*]}}$. Since S is closed and $\text{dom } S$ is closed, also $\text{dom } S^{[*]}$ is closed by Proposition 2.4. Hence $f \in \text{dom } S^{[*]}$ and $\{f, f'\} \in S^{[*]}$ for some $f' \in \mathfrak{K}_1$. Now it suffices to show that $\{f', g\} \in X^{[*]}$, because then $\{f, g\} \in X^{[*]}S^{[*]}$. Indeed, for each $\{h, u\} \in X$ there is $u' \in \mathfrak{K}_2$ such that $\{u, u'\} \in S$, due to the condition $\text{ran } X \subset \text{dom } S$. Then for all $\{f, f'\} \in S^{[*]}$ one has

$$(2.11) \quad [g, h] - [f', u] = [g, h] - [f, u'].$$

Clearly, $\{h, u'\} \in SX$ and thus (2.10) implies that $[g, h] = [f', u]$ for all $\{h, u\} \in X$. This means that $\{f', g\} \in X^{[*]}$. Thus $(SX)^{[*]} \subset X^{[*]}S^{[*]}$.

(ii) This statement is obtained by applying part (i) to the inverse $(YS)^{-1} = S^{-1}Y^{-1}$. \square

The following theorem concerns the composition of two unitary relations; the results therein will be important in the sequel.

Theorem 2.13. *Let \mathfrak{K}_1 , \mathfrak{K}_2 , and \mathfrak{K}_3 be Krein spaces and let the linear relations $T : \mathfrak{K}_1 \rightarrow \mathfrak{K}_2$ and $S : \mathfrak{K}_2 \rightarrow \mathfrak{K}_3$ be isometric. Then:*

(i) *the linear relation $ST : \mathfrak{K}_1 \rightarrow \mathfrak{K}_3$ is isometric.*

In addition, let the linear relations $T : \mathfrak{K}_1 \rightarrow \mathfrak{K}_2$ and $S : \mathfrak{K}_2 \rightarrow \mathfrak{K}_3$ be unitary. Then:

- (ii) *if $\text{dom } S$ and $T \hat{+} (\{0\} \times \ker S)$ are closed then $ST : \mathfrak{K}_1 \rightarrow \mathfrak{K}_3$ is unitary;*
- (iii) *if*

$$(2.12) \quad \text{ran } T \subset \text{dom } S \quad \text{and} \quad \text{dom } S \text{ is closed,}$$

then $ST : \mathfrak{K}_1 \rightarrow \mathfrak{K}_3$ is unitary and $\text{dom } ST = \text{dom } T$;

(iv) *if $\text{dom } T$ and $S \hat{+} (\text{mul } T \times \{0\})$ are closed then $ST : \mathfrak{K}_1 \rightarrow \mathfrak{K}_3$ is unitary;*

(v) *if*

$$(2.13) \quad \text{ran } T \supset \text{dom } S \quad \text{and} \quad \text{dom } T \text{ is closed,}$$

then $ST : \mathfrak{K}_1 \rightarrow \mathfrak{K}_3$ is unitary and $\text{ran } ST = \text{ran } S$;

(vi) *if $\text{ran } T = \text{dom } S$ and $\text{ran } S = \mathfrak{K}_3$, then the unitary relation $ST : \mathfrak{K}_1 \rightarrow \mathfrak{K}_3$ is bounded and single-valued (not necessarily densely defined);*

(vii) *if $T \in [\mathfrak{K}_1, \mathfrak{K}_2]$ or $S \in [\mathfrak{K}_2, \mathfrak{K}_3]$, then $ST : \mathfrak{K}_1 \rightarrow \mathfrak{K}_3$ is unitary;*

(viii) *if $T \in [\mathfrak{K}_1, \mathfrak{K}_2]$ and $S \in [\mathfrak{K}_2, \mathfrak{K}_3]$, then ST is a unitary operator which belongs to $[\mathfrak{K}_1, \mathfrak{K}_3]$.*

Proof. (i) Since S and T are isometric, one has $S^{-1} \subset S^{[*]}$ and $T^{-1} \subset T^{[*]}$. The definition of the product of relations implies that $T^{-1}S^{-1} \subset T^{[*]}S^{[*]}$. Lemma 2.2 yields

$$(2.14) \quad (ST)^{-1} = T^{-1}S^{-1} \subset T^{[*]}S^{[*]} \subset (ST)^{[*]}.$$

Hence, the relation ST is isometric.

(ii) Since S and T are unitary, ST is isometric by part (i), i.e., $(ST)^{-1} \subset (ST)^{[*]}$. To see that ST is unitary it suffices to prove the inclusion $(ST)^{[*]} \subset (ST)^{-1} = T^{[*]}S^{[*]}$ (where the last identity is due to S and T being unitary). The linear relation T_0 defined by

$$T_0 := T \cap (\mathfrak{H}_1 \times \text{dom } S) = \{ \{h, h'\} \in T : h' \in \text{dom } S \}$$

satisfies the inclusion $\text{ran } T_0 \subset \text{dom } S$. Hence from Lemma 2.12 one obtains

$$(ST)^{[*]} \subset (ST_0)^{[*]} = T_0^{[*]}S^{[*]}.$$

Now it is enough to prove that $T_0^{[*]}S^{[*]} \subset T^{[*]}S^{[*]}$ (then also $T_0^{[*]}S^{[*]} = T^{[*]}S^{[*]}$ holds). Since T is unitary, it follows from the assumptions in (ii) that

$$(2.15) \quad T_0^{[*]} = T^{[*]} \hat{+} (\ker S \times \{0\}).$$

Now let $\{f, g\} \in T_0^{[*]}S^{[*]}$. Then for some $f' \in \mathfrak{K}_2$ one has $\{f, f'\} \in S^{[*]}$ and $\{f', g\} \in T_0^{[*]}$. Hence due to (2.15) $\{f' - f_0, g\} \in T^{[*]}$ for some $f_0 \in \ker S$. Since S is unitary one has $f_0 \in \text{mul } S^{[*]} (= \ker S)$. Thus $\{f, f' - f_0\} \in S^{[*]}$ and therefore $\{f, g\} \in T^{[*]}S^{[*]}$. This completes the proof of part (ii).

(iii) By the assumptions in (2.12) one obtains the statement directly from Lemma 2.12: $(ST)^{[*]} = T^{[*]}S^{[*]} = T^{-1}S^{-1} = (ST)^{-1}$. The equality $\text{dom } ST = \text{dom } T$ is clear due to the assumption $\text{ran } T \subset \text{dom } S$.

(iv) This statement is obtained by applying part (ii) to the inverse $(ST)^{-1} = T^{-1}S^{-1}$ and by taking into account Lemma 2.6 and the equivalence stated in (i) of Proposition 2.7.

(v) This is again an immediate consequence of Lemma 2.12; it can be obtained also from (iii) by means of inverses.

(vi) If $\text{ran } T = \text{dom } S$ and $\text{ran } S = \mathfrak{K}_3$, then $\text{dom } S$ and $\text{dom } T$ are closed by (i) of Proposition 2.7. Therefore, by part (v), the relation $ST : \mathfrak{K}_1 \rightarrow \mathfrak{K}_3$ is unitary and $\text{ran } ST = \text{ran } S = \mathfrak{K}_3$. Furthermore, ST bounded and single-valued by (iii) of Lemma 2.8.

(vii) The relations S and T are assumed to be unitary. If in addition $S \in [\mathfrak{K}_2, \mathfrak{K}_3]$, then by definition $\text{dom } S = \mathfrak{K}_2$ and moreover $\ker S = \{0\}$ by Proposition 2.7. Hence the relation ST is unitary by part (ii). On the other hand, if $T \in [\mathfrak{K}_1, \mathfrak{K}_2]$, then $\text{dom } T = \mathfrak{K}_1$, $\text{ran } T = \mathfrak{K}_2$, and now part (v) shows that ST is unitary.

(viii) This is clear and a well-known fact. \square

Observe that in Theorem 2.13 the only standard result in the literature is the last statement (viii). Notice also that (iii) is in fact a special case of (ii). Indeed, if $\text{ran } T \subset \text{dom } S$ then $\text{mul } T \supset \ker S$ by Proposition 2.7 and hence in this case $T \widehat{+} (\{0\} \times \ker S) = T$ is closed. Likewise (v) is a special case of (iv).

Corollary 2.14. *Let the linear relations $T : \mathfrak{K}_1 \rightarrow \mathfrak{K}_2$ and $S : \mathfrak{K}_2 \rightarrow \mathfrak{K}_3$ be unitary. Then:*

- (i) *if $\text{dom } S$ is closed and $\dim \ker S < \infty$ then $ST : \mathfrak{K}_1 \rightarrow \mathfrak{K}_3$ is unitary;*
- (ii) *if $\text{dom } T$ is closed and $\dim \text{mul } T < \infty$ then $ST : \mathfrak{K}_1 \rightarrow \mathfrak{K}_3$ is unitary;*
- (iii) *if $\dim \mathfrak{K}_2 < \infty$ then $ST : \mathfrak{K}_1 \rightarrow \mathfrak{K}_3$ is unitary.*

Proof. The statements (i) and (ii) are immediate consequences of parts (ii) and (iv) in Theorem 2.13, respectively.

As to (iii) observe that if \mathfrak{K}_2 is finite-dimensional then automatically the assumptions in (i) and (ii) are satisfied. \square

The following examples show that in the case of infinite dimensional spaces unitary operators may be unbounded and their set does not form a semigroup, that is, the product of two unitary operators need not be a unitary operator.

Example 2.15. Let K be a densely defined operator on a Hilbert space \mathfrak{H} and define the block operator matrix T by

$$(2.16) \quad T = \begin{pmatrix} I_{\mathfrak{H}} & K \\ 0 & I_{\mathfrak{H}} \end{pmatrix}.$$

Then T is an injective operator, i.e., $\ker T = \{0\}$, $\text{mul } T = \{0\}$. It is easy to see that T is closed if and only if K is closed. The inverse of T is given by

$$(2.17) \quad T^{-1} = \begin{pmatrix} I_{\mathfrak{H}} & -K \\ 0 & I_{\mathfrak{H}} \end{pmatrix}$$

and hence T is densely defined with dense range; in fact $\text{dom } T = \text{ran } T = \mathfrak{H} \oplus \text{dom } K$. Now consider $\mathfrak{H} \oplus \mathfrak{H}$ as the Krein space $(\mathfrak{H}^2, J_{\mathfrak{H}})$ with the fundamental symmetry

$$(2.18) \quad J_{\mathfrak{H}} := \begin{pmatrix} 0 & -iI_{\mathfrak{H}} \\ iI_{\mathfrak{H}} & 0 \end{pmatrix}.$$

Then

$$(2.19) \quad T^{[*]} = \begin{pmatrix} I_{\mathfrak{H}} & -K^* \\ 0 & I_{\mathfrak{H}} \end{pmatrix}.$$

The identities (2.17) and (2.19) show that T is isometric (unitary) if and only if K is symmetric (resp. selfadjoint). Therefore, if K_1, K_2 are two unbounded selfadjoint operators in \mathfrak{H} such that $K_1 + K_2$ is not selfadjoint, the product $T_1 T_2$ of the unitary operators T_1 and T_2 ,

$$T_1 T_2 = \begin{pmatrix} I_{\mathfrak{H}} & K_1 \\ 0 & I_{\mathfrak{H}} \end{pmatrix} \begin{pmatrix} I_{\mathfrak{H}} & K_2 \\ 0 & I_{\mathfrak{H}} \end{pmatrix} = \begin{pmatrix} I_{\mathfrak{H}} & K_1 + K_2 \\ 0 & I_{\mathfrak{H}} \end{pmatrix},$$

is not a unitary operator in $(\mathfrak{H}^2, J_{\mathfrak{H}})$. Here both assumptions in (2.12) can fail to hold. This is the case if, for instance, K_1 and K_2 are selfadjoint operators in \mathfrak{H} such that $\text{dom } K_1 \cap \text{dom } K_2 = \{0\}$.

Note also that if K_1 is an unbounded selfadjoint operator in \mathfrak{H} and $K_2 = -K_1$ then $\text{ran } T_2 = \text{dom } T_1$, cf. (2.17), so that $\text{dom } T_1 T_2 = \text{dom } T_2$. Now the product $T_1 T_2$ is not closed and hence it cannot be unitary. In this case the first assumption in (2.12) and (2.13) is satisfied, while the second assumption in (2.12), (2.13) fails to hold. The second assumption in (ii) and (iv) of Theorem 2.13 is also satisfied, since $\ker T_1 = \{0\}$ and $\text{mul } T_2 = \{0\}$.

Obviously, ST can be unitary even if the assumptions (2.12) and (2.13) are not satisfied. Also only one of the two conditions in (2.12), (2.13) or in (ii), (iv) of Theorem 2.13 is not sufficient for the product ST to be unitary.

Example 2.16. Let K_2 be a selfadjoint operator in \mathfrak{H} . Then the linear relation T_2 given by

$$T_2 = \left\{ \left\{ \begin{pmatrix} K_2 h \\ h \end{pmatrix}, \begin{pmatrix} 0 \\ g \end{pmatrix} \right\} : h \in \text{dom } K_2, g \in \mathfrak{H} \right\}$$

is unitary in $(\mathfrak{H}^2, J_{\mathfrak{H}})$ with $\text{dom } T_2 = \ker T_2 = (\text{gr } K_2)^{-1}$, $\text{ran } T_2 = \text{mul } T_2 = \{0\} \times \mathfrak{H} (\subset \mathfrak{H} \times \mathfrak{H})$ closed. If T_1 is as in (2.16) with K_1 a selfadjoint operator then the product $T_1 T_2$ is unitary. Here $\text{dom } T_1$ is closed if and only if K_1 is bounded, in which case the assumptions in (2.12) are satisfied. However, if K_1 is unbounded then both of the assumptions in (2.12) fail to hold and also the first assumption in (2.13) is not satisfied. It is not difficult to check that both assumptions in (iv) of Theorem 2.13 are satisfied.

The product $T_2 T_1$ is given by

$$T_2 T_1 = \left\{ \left\{ \begin{pmatrix} (K_2 - K_1)h \\ h \end{pmatrix}, \begin{pmatrix} 0 \\ g \end{pmatrix} \right\} : h \in \text{dom } K_2 \cap \text{dom } K_1, g \in \mathfrak{H} \right\}.$$

This relation is unitary if and only if $K_2 - K_1$ is selfadjoint. Now the second assumption in (2.12) is satisfied, while the first assumption in (2.12) does not hold. If, for instance, $\text{dom } K_1 \cap \text{dom } K_2 = \{0\}$, then $T_2 T_1$ is not unitary. In this case both assumptions in (2.13) fail to hold. On the other hand, if K_1 is bounded then the assumptions in (2.13) are satisfied and $T_2 T_1$ is unitary. The product $T_2 T_1$ is also unitary if K_2 is bounded, while both of the assumptions in (2.13) fail to hold if K_1 is unbounded.

The first assumption in (ii) of Theorem 2.13 holds. The second assumption in (ii) of Theorem 2.13 is equivalent for the row operator $(K_1 K_2)$ to be closed, which therefore by part (ii) implies that $K_2 - K_1$ is selfadjoint. Obviously, $K_2 - K_1$ can be selfadjoint even if the row operator $(K_1 K_2)$ is not closed: consider e.g. $-K_1 = K_2 =: K$: here $K_2 - K_1 = 2K$ is selfadjoint, but the row operator $(K_1 K_2) = (-K K)$ is not closed if K is unbounded: let

$h_n \rightarrow h \notin \text{dom } K$. Then $\text{col}(h_n, h_n) \rightarrow \text{col}(h, h)$ and $(-K K)\text{col}(h_n, h_n) \equiv 0$, and closedness would imply that $h \in \text{dom } K$.

Also observe that the linear relation $T_2^2 = T_2$ is unitary in $(\mathfrak{H}^2, J_{\mathfrak{H}})$. However, $\text{dom } T_2 \cap \text{ran } T_2 = \{0\}$, if $0 \notin \sigma_p(K_2)$ and in this case $\text{dom } T_2$ and $\text{ran } T_2$ are closed.

2.3. The main transform. It is convenient to interpret the Hilbert space $\mathfrak{H}^2 = \mathfrak{H} \oplus \mathfrak{H}$ as a Kreĭn space $(\mathfrak{H}^2, J_{\mathfrak{H}})$ whose inner product is determined by the fundamental symmetry $J_{\mathfrak{H}}$ of the form (2.18); notice the connection to the definition of the adjoint of linear relations in (2.1). There is a useful and important transform which gives a connection between the subspaces of a Hilbert space $\mathfrak{H} \oplus \mathcal{H}$ and linear relations from the Kreĭn space $(\mathfrak{H}^2, J_{\mathfrak{H}})$ to the Kreĭn space $(\mathcal{H}^2, J_{\mathcal{H}})$, which will be now recalled from [15]. Let \mathfrak{H} and \mathcal{H} be Hilbert spaces and let their Cartesian product be denoted by $\tilde{\mathfrak{H}} = \mathfrak{H} \oplus \mathcal{H}$. Define the linear mapping \mathcal{J} from $\mathfrak{H}^2 \times \mathcal{H}^2$ to $(\mathfrak{H} \oplus \mathcal{H})^2$ by

$$\mathcal{J} : \left\{ \begin{pmatrix} f \\ f' \end{pmatrix}, \begin{pmatrix} h \\ h' \end{pmatrix} \right\} \mapsto \left\{ \begin{pmatrix} f \\ h \end{pmatrix}, \begin{pmatrix} f' \\ -h' \end{pmatrix} \right\}, \quad f, f' \in \mathfrak{H}, h, h' \in \mathcal{H}.$$

This mapping establishes a one-to-one correspondence between the (closed) linear relations $\Gamma : \mathfrak{H}^2 \rightarrow \mathcal{H}^2$ and the (closed) linear relations $\tilde{\Gamma}$ in $\tilde{\mathfrak{H}} = \mathfrak{H} \oplus \mathcal{H}$ via

$$(2.20) \quad \Gamma \mapsto \tilde{\Gamma} := \mathcal{J}(\Gamma) = \left\{ \left\{ \begin{pmatrix} f \\ h \end{pmatrix}, \begin{pmatrix} f' \\ -h' \end{pmatrix} \right\} : \left\{ \begin{pmatrix} f \\ f' \end{pmatrix}, \begin{pmatrix} h \\ h' \end{pmatrix} \right\} \in \Gamma \right\}.$$

The mapping \mathcal{J} plays a principal role and it is referred to as the *main transform*. Some basic properties of this transform are stated in the following proposition.

Proposition 2.17. *Let the linear relation Γ from $(\mathfrak{H}^2, J_{\mathfrak{H}})$ to $(\mathcal{H}^2, J_{\mathcal{H}})$ and the linear relation $\tilde{\Gamma}$ in $\mathfrak{H} \oplus \mathcal{H}$ be connected by $\tilde{\Gamma} = \mathcal{J}(\Gamma)$. The main transform \mathcal{J} establishes a one-to-one correspondence between the contractive, isometric, and unitary relations Γ from $(\mathfrak{H}^2, J_{\mathfrak{H}})$ to $(\mathcal{H}^2, J_{\mathcal{H}})$ and the dissipative, symmetric, and selfadjoint relations $\tilde{\Gamma}$ in $\mathfrak{H} \oplus \mathcal{H}$, respectively.*

2.4. Nevanlinna families. A family of linear relations $M(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, in a Hilbert space \mathcal{H} is called a *Nevanlinna family* if:

- (i) for every $\lambda \in \mathbb{C}_+(\mathbb{C}_-)$ the relation $M(\lambda)$ is maximal dissipative (resp. accumulative);
- (ii) $M(\lambda)^* = M(\bar{\lambda})$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$;
- (iii) for some, and hence for all, $\mu \in \mathbb{C}_+(\mathbb{C}_-)$ the operator family $(M(\lambda) + \mu)^{-1}(\in [\mathcal{H}])$ is holomorphic for all $\lambda \in \mathbb{C}_+(\mathbb{C}_-)$.

By the maximality condition, each relation $M(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, is necessarily closed. The *class of all Nevanlinna families* in a Hilbert space is denoted by $\tilde{R}(\mathcal{H})$. If the multi-valued part $\text{mul } M(\lambda)$ of $M(\cdot) \in \tilde{R}(\mathcal{H})$ is nontrivial, then it is independent of $\lambda \in \mathbb{C} \setminus \mathbb{R}$, so that

$$(2.21) \quad M(\lambda) = M_s(\lambda) \oplus M_{\infty}, \quad M_{\infty} = \{0\} \times \text{mul } M(\lambda), \quad \lambda \in \mathbb{C} \setminus \mathbb{R},$$

where $M_s(\lambda)$ is a Nevanlinna family of densely defined operators in $\mathcal{H} \ominus \text{mul } M(\lambda)$, [34].

Clearly, if $M(\cdot) \in \tilde{R}(\mathcal{H})$, then $M_{\infty} \subset M(\lambda) \cap M(\lambda)^*$ for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$. The following *subclasses* of the class $\tilde{R}(\mathcal{H})$ will be useful:

- $R(\mathcal{H})$ is the set of all $M(\cdot) \in \tilde{R}(\mathcal{H})$ for which $\text{mul } M(\lambda) = \{0\}$;
- $R^s(\mathcal{H})$ is the set of all $M(\cdot) \in \tilde{R}(\mathcal{H})$ for which $M(\lambda) \cap M(\lambda)^* = \{0\}$ for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$;
- $R^u(\mathcal{H})$ is the set of all $M(\cdot) \in \tilde{R}(\mathcal{H})$ for which $M(\lambda) \widehat{+} M(\lambda)^* = \mathcal{H}^2$ for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$;
- $R[\mathcal{H}]$ is the set of all $M(\cdot) \in \tilde{R}[\mathcal{H}]$ for which $\text{dom } M(\lambda) = \mathcal{H}$ for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$;

$R^s[\mathcal{H}]$ is the set of all $M(\cdot) \in R[\mathcal{H}]$ for which $\ker \text{Im } M(\lambda) = \{0\}$ for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$;

$R^u[\mathcal{H}]$ is the set of all $M(\cdot) \in R^s[\mathcal{H}]$ for which $0 \in \rho(\text{Im } M(\lambda))$ for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$;

$\tilde{R}^c(\mathcal{H})$ is the set of all constant Nevanlinna families.

The subclasses of $\tilde{R}(\mathcal{H})$ can be equivalently defined by assuming the corresponding property of $M(\lambda)$ only at a single point $\lambda \in \mathbb{C} \setminus \mathbb{R}$, [15]. Moreover, it is easy to show that $R^u[\mathcal{H}] = R^u(\mathcal{H})$, see [15]. The Nevanlinna functions in $R^s[\mathcal{H}]$ and $R^u[\mathcal{H}]$ will be called *strict* and *uniformly strict*, respectively.

If $M(\cdot) \in R[\mathcal{H}]$, then it admits the following integral representation

$$(2.22) \quad M(\lambda) = A + B\lambda + \int_{\mathbb{R}} \left(\frac{1}{t - \lambda} - \frac{t}{t^2 + 1} \right) d\Sigma(t), \quad \int_{\mathbb{R}} \frac{d\Sigma(t)}{t^2 + 1} \in [\mathcal{H}],$$

where $A = A^* \in [\mathcal{H}]$, $0 \leq B = B^* \in [\mathcal{H}]$, the $[\mathcal{H}]$ -valued family $\Sigma(\cdot)$ is nondecreasing, and the integral is uniformly convergent in the strong topology, cf. [7], [28].

A pair $\{\Phi, \Psi\}$ of holomorphic $[\mathcal{H}]$ -valued functions on $\mathbb{C}_+ \cup \mathbb{C}_-$ is said to be a *Nevanlinna pair* if:

- (N1) $\text{Im } \Phi(\lambda)^* \Psi(\lambda) / \text{Im } \lambda \geq 0$, $\lambda \in \mathbb{C}_+ \cup \mathbb{C}_-$;
- (N2) $\Psi(\bar{\lambda})^* \Phi(\lambda) - \Phi(\bar{\lambda})^* \Psi(\lambda) = 0$, $\lambda \in \mathbb{C}_+ \cup \mathbb{C}_-$;
- (N3) $0 \in \rho(\Psi(\lambda) \pm i\Phi(\lambda))$, $\lambda \in \mathbb{C}_\pm$.

Two Nevanlinna pairs $\{\Phi_1, \Psi_1\}$ and $\{\Phi_2, \Psi_2\}$ are said to be equivalent, if $\Phi_2(\lambda) = \Phi_1(\lambda)\chi(\lambda)$ and $\Psi_2(\lambda) = \Psi_1(\lambda)\chi(\lambda)$ for some operator function $\chi(\lambda) \in [\mathcal{H}]$, which is holomorphic and invertible on $\mathbb{C}_+ \cup \mathbb{C}_-$. If $\{\Phi, \Psi\}$ is a Nevanlinna pair, then the following kernel is nonnegative on $\mathbb{C}_+ \cup \mathbb{C}_-$:

$$(2.23) \quad N_{\Phi\Psi}(\lambda, \mu) = \frac{\Phi(\mu)^* \Psi(\lambda) - \Psi(\mu)^* \Phi(\lambda)}{\lambda - \bar{\mu}}, \quad \lambda, \mu \in \mathbb{C}_+ \cup \mathbb{C}_-.$$

The set of Nevanlinna families $\tau(\lambda)$ and the set of equivalence classes of Nevanlinna pairs $\{\Phi, \Psi\}$ are in a one-to-one correspondence via the formula

$$(2.24) \quad \tau(\lambda) = \{\Phi(\lambda), \Psi(\lambda)\} := \{\{\Phi(\lambda)h, \Psi(\lambda)h\} : h \in \mathcal{H}\}.$$

Moreover, strict and uniformly strict Nevanlinna families are characterized by the conditions $0 \notin \sigma_p(N_{\Phi\Psi}(\lambda, \lambda))$ and $0 \in \rho(N_{\Phi\Psi}(\lambda, \lambda))$ for some $\lambda \in \mathbb{C} \setminus \mathbb{R}$, respectively.

2.5. Shmul'yan transform of linear relations. Let \mathcal{H} and \mathcal{K} be Hilbert spaces and let W be a linear relation from the Hilbert space $\mathfrak{H}^2 = \mathcal{H} \oplus \mathcal{H}$ to the Hilbert space $\mathcal{K}^2 = \mathcal{K} \oplus \mathcal{K}$. For any linear relation Θ in \mathcal{H} ,

$$(2.25) \quad W[\Theta] = \{\hat{k} \in \mathcal{K}^2 : \{\hat{h}, \hat{k}\} \in W, \hat{h} \in \Theta\},$$

defines a linear relation $W[\Theta]$ in \mathcal{K} .

Definition 2.18. The linear relation $W[\Theta]$ in \mathcal{K} , defined by (2.25), is said to be the *Shmul'yan transform* of the linear relation Θ in \mathcal{H} , induced by the linear relation $W : \mathcal{H}^2 \rightarrow \mathcal{K}^2$.

For any pair of relations Θ_1 and Θ_2 in the Hilbert space \mathcal{H} , there are the inclusions

$$W[\Theta_1 \cap \Theta_2] \subset W[\Theta_1] \cap W[\Theta_2],$$

and

$$(2.26) \quad W[\Theta_1] \hat{+} W[\Theta_2] \subset W[\Theta_1 \hat{+} \Theta_2].$$

Furthermore, if $\ker W = \{0\}$, then

$$(2.27) \quad W[\Theta_1 \cap \Theta_2] = W[\Theta_1] \cap W[\Theta_2],$$

and, if $\text{mul } W = \{0\}$, then

$$(2.28) \quad W[\Theta_1] \widehat{+} W[\Theta_2] = W[\Theta_1 \widehat{+} \Theta_2].$$

Now interpret W as a linear relation from the Kreĭn space $(\mathcal{H}^2, J_{\mathcal{H}})$ to the Kreĭn space $(\mathcal{K}^2, J_{\mathcal{K}})$, where the inner products are defined as in (2.18).

Lemma 2.19. *Let W be a linear relation from the Kreĭn space $(\mathcal{H}^2, J_{\mathcal{H}})$ to the Kreĭn space $(\mathcal{K}^2, J_{\mathcal{K}})$. Then $\{\widehat{\alpha}, \widehat{\beta}\} \in W^{[*]}$ if and only if*

$$(2.29) \quad [\widehat{\beta}, \widehat{h}]_{\mathcal{H}^2} = [\widehat{\alpha}, \widehat{k}]_{\mathcal{K}^2} \quad \text{for all } \{\widehat{h}, \widehat{k}\} \in W.$$

Let Θ be a linear relation in \mathcal{H} and let $\{\widehat{\alpha}, \widehat{\beta}\} \in W^{[*]}$. Then

$$(2.30) \quad \widehat{\beta} \in \Theta^* \Leftrightarrow \widehat{\alpha} \in W[\Theta]^*.$$

Proof. Since $\{\widehat{\alpha}, \widehat{\beta}\} \in W^{[*]}$, the identity (2.29) is satisfied for all $\{\widehat{h}, \widehat{k}\} \in W$. The equivalence in (2.30) is a straightforward consequence of (2.29) and the connection of the inner products in the Kreĭn spaces $(\mathcal{H}^2, J_{\mathcal{H}})$ and $(\mathcal{K}^2, J_{\mathcal{K}})$ to the definition of adjoint in (2.1). \square

Corollary 2.20. *Let W be an isometric linear relation from the Kreĭn space $(\mathcal{H}^2, J_{\mathcal{H}})$ to the Kreĭn space $(\mathcal{K}^2, J_{\mathcal{K}})$ and let Θ be a linear relation in \mathcal{H} . Then*

$$W[\Theta^*] \subset W[\Theta]^*.$$

Moreover, Θ is dissipative (symmetric) if and only if $W[\Theta]$ is dissipative (symmetric).

Proof. Let $\widehat{k} \in W[\Theta]$ and $\widehat{\alpha} \in W[\Theta]^*$. Then there exist elements $\widehat{h} \in \Theta$ and $\widehat{\beta} \in \Theta^*$ such that $\{\widehat{h}, \widehat{k}\} \in W$ and $\{\widehat{\beta}, \widehat{\alpha}\} \in W$. Since $\{\widehat{\alpha}, \widehat{\beta}\} \in W^{-1} = W^{[*]}$, it follows from Lemma 2.19 that

$$[\widehat{k}, \widehat{\alpha}]_{\mathcal{K}^2} = [\widehat{h}, \widehat{\beta}]_{\mathcal{H}^2} = 0.$$

This shows that $W[\Theta]$ and $W[\Theta]^*$ are orthogonal in the Kreĭn space $(\mathcal{K}^2, J_{\mathcal{K}})$.

For every $\{\widehat{h}, \widehat{k}\} \in W$ one has $\{\widehat{k}, \widehat{h}\} \in W^{-1} \subset W^{[*]}$, since W is isometric. Therefore,

$$(2.31) \quad 0 = [\widehat{h}, \widehat{h}]_{\mathcal{H}^2} - [\widehat{k}, \widehat{k}]_{\mathcal{K}^2} = 2i [\text{Im}(h', h) - \text{Im}(k', k)], \quad \{\widehat{h}, \widehat{k}\} \in W,$$

where the identity on the left is due to (2.29), and the identity on the right is due to the definition of the inner products. Note that if $\widehat{h} \in \Theta$, then there exists $\widehat{k} \in W[\Theta]$ with $\{\widehat{h}, \widehat{k}\} \in W$. Hence, if $W[\Theta]$ is dissipative or symmetric, then (2.31) shows that Θ is dissipative or symmetric, respectively. Conversely, if $\widehat{k} \in W[\Theta]$, then there exists $\widehat{h} \in \Theta$ with $\{\widehat{h}, \widehat{k}\} \in W$. Hence, if Θ is dissipative or symmetric, then so is $W[\Theta]$. \square

In the general context of Corollary 2.20 it seems difficult to conclude anything about the maximality of the dissipative (symmetric) relations. Of course, when W is a standard unitary operator, then some known properties can be easily recovered, cf. [36],[44].

Corollary 2.21. *Let W be a standard unitary operator from the Kreĭn space $(\mathcal{H}^2, J_{\mathcal{H}})$ onto the Kreĭn space $(\mathcal{K}^2, J_{\mathcal{K}})$. Let Θ be a linear relation in \mathcal{H} . Then:*

$$W[\Theta^*] = W[\Theta]^*.$$

Furthermore,

- (i) Θ is maximal dissipative $\Leftrightarrow W[\Theta]$ is maximal dissipative;
- (ii) Θ is maximal symmetric $\Leftrightarrow W[\Theta]$ is maximal symmetric;
- (iii) Θ is selfadjoint $\Leftrightarrow W[\Theta]$ is selfadjoint.

In case W is a standard unitary operator from \mathcal{H}^2 onto \mathcal{K}^2 , the Shmul'yan transform is usually written out in components. Then W is bounded and with bounded inverse and it can be represented in the block form

$$(2.32) \quad W = \begin{pmatrix} W_{00} & W_{01} \\ W_{10} & W_{11} \end{pmatrix}, \quad W_{ij} \in [\mathcal{H}, \mathcal{K}], \quad i, j = 0, 1.$$

If Θ is a linear relation in \mathcal{H} , then $W[\Theta]$ in (2.25) takes the form

$$(2.33) \quad W[\Theta] = \{ \{W_{00}h + W_{01}h', W_{10}h + W_{11}h'\} : \{h, h'\} \in \Theta \}.$$

Clearly, $W[\Theta]$ is contained in the linear relation

$$(2.34) \quad (W_{10} + W_{11}\Theta)(W_{00} + W_{01}\Theta)^{-1} = \{W_{00}h + W_{01}h', W_{10}h + W_{11}h''\} : \{h, h'\}, \{h, h''\} \in \Theta \}.$$

In fact, the following equality holds

$$(2.35) \quad (W_{10} + W_{11}\Theta)(W_{00} + W_{01}\Theta)^{-1} = W[\Theta] \widehat{+} \{0, W_{11}(\text{mul } \Theta)\}.$$

Hence, if Θ is a relation with $W_{11}(\text{mul } \Theta) = \{0\}$, and in particular if Θ is an operator, the linear relations in (2.33) and in (2.34) coincide.

3. BOUNDARY RELATIONS AND WEYL FAMILIES

3.1. Definitions and basic properties. Let S be a closed symmetric linear relation in the Hilbert space \mathfrak{H} . It is not assumed that the defect numbers of S are equal or finite. A boundary relation for S^* is defined as follows.

Definition 3.1. Let S be a closed symmetric linear relation in a Hilbert space \mathfrak{H} and let \mathcal{H} be an auxiliary Hilbert space. A linear relation $\Gamma : \mathfrak{H}^2 \mapsto \mathcal{H}^2$ is called a *boundary relation* for S^* , if:

(G1) $\text{dom } \Gamma$ is dense in S^* and the identity

$$(3.1) \quad (f', g)_{\mathfrak{H}} - (f, g')_{\mathfrak{H}} = (h', k)_{\mathcal{H}} - (h, k')_{\mathcal{H}},$$

holds for every $\{\widehat{f}, \widehat{h}\}, \{\widehat{g}, \widehat{k}\} \in \Gamma$;

(G2) Γ is maximal in the sense that if $\{\widehat{g}, \widehat{k}\} \in \mathfrak{H}^2 \times \mathcal{H}^2$ satisfies (3.1) for every $\{\widehat{f}, \widehat{h}\} \in \Gamma$, then $\{\widehat{g}, \widehat{k}\} \in \Gamma$.

Here $\widehat{f} = \{f, f'\}, \widehat{g} = \{g, g'\} \in \text{dom } \Gamma (\subset \mathfrak{H}^2), \widehat{h} = \{h, h'\}, \widehat{k} = \{k, k'\} \in \text{ran } \Gamma (\subset \mathcal{H}^2)$.

The condition (3.1) in (G1) can be interpreted as an abstract Green's identity. Using the terminology of Kreĭn spaces (3.1) means that Γ is an isometric relation from the Kreĭn space $(\mathfrak{H}^2, J_{\mathfrak{H}})$ to the Kreĭn space $(\mathcal{H}^2, J_{\mathcal{H}})$, since

$$(3.2) \quad (J_{\mathfrak{H}}\widehat{f}, \widehat{g})_{\mathfrak{H}^2} = (J_{\mathcal{H}}\widehat{h}, \widehat{k})_{\mathcal{H}^2}, \quad \{\widehat{f}, \widehat{h}\}, \quad \{\widehat{g}, \widehat{k}\} \in \Gamma.$$

The maximality condition (G2) and Proposition 2.7 now yield the following result.

Proposition 3.2. ([15]) *Let \mathfrak{H} and \mathcal{H} be Hilbert spaces and let S be a closed symmetric linear relation in \mathfrak{H} . Then a linear relation $\Gamma : \mathfrak{H}^2 \mapsto \mathcal{H}^2$ is a boundary relation for S^* if and only if Γ is a unitary relation from the Kreĭn space $(\mathfrak{H}^2, J_{\mathfrak{H}})$ to the Kreĭn space $(\mathcal{H}^2, J_{\mathcal{H}})$ with $S = \ker \Gamma$.*

In some cases the following criterion for a linear relation to be a boundary relation is useful; see [15, Proposition 3.6].

Proposition 3.3. *The linear relation $\Gamma : \mathfrak{H}^2 \mapsto \mathcal{H}^2$ is a boundary relation for S^* if and only if the following conditions hold:*

- (i) $\text{dom } \Gamma$ is dense in S^* ;
- (ii) Γ is closed and isometric from the Krein space $(\mathfrak{H}^2, J_{\mathfrak{H}})$ to the Krein space $(\mathcal{H}^2, J_{\mathcal{H}})$;
- (iii) $\text{ran}(\Gamma(\widehat{\mathfrak{N}}_{\lambda}(T)) + \lambda) = \mathcal{H}$ for some (and, hence, for all) $\lambda \in \mathbb{C}_+$ and for some (and, hence, for all) $\lambda \in \mathbb{C}_-$.

Note that a boundary relation Γ is automatically closed and linear, since it is a unitary relation from the Krein space $(\mathfrak{H}^2, J_{\mathfrak{H}})$ to the Krein space $(\mathcal{H}^2, J_{\mathcal{H}})$. Observe that the inverse $\Gamma^{-1} : (\mathcal{H}^2, J_{\mathcal{H}}) \rightarrow (\mathfrak{H}^2, J_{\mathfrak{H}})$ is also unitary; see Lemma 2.6. Therefore, in this case Γ^{-1} can be interpreted as a boundary relation for $\widetilde{S}^* \subset \mathcal{H}^2$, the adjoint of the closed symmetric relation

$$(3.3) \quad \widetilde{S} := \ker \Gamma^{-1} = \text{mul } \Gamma (\subset \mathcal{H}^2).$$

Let Γ be a boundary relation for S^* and let $T = \text{dom } \Gamma$. According to [15, Proposition 2.12] the linear relation T in \mathfrak{H} satisfies

$$(3.4) \quad S \subset T \subset S^*, \quad \text{clos } T = S^*.$$

The eigenspaces $\mathfrak{N}_{\lambda}(T)$ and $\widehat{\mathfrak{N}}_{\lambda}(T)$ for T are defined by

$$(3.5) \quad \mathfrak{N}_{\lambda}(T) = \ker(T - \lambda), \quad \widehat{\mathfrak{N}}_{\lambda}(T) = \{ \{f, \lambda f\} \in T : f \in \mathfrak{N}_{\lambda}(T) \}.$$

For notational convenience the usual defect spaces of S are denoted here by $\mathfrak{N}_{\lambda}(S^*)$ and $\widehat{\mathfrak{N}}_{\lambda}(S^*)$.

For all elements $\{\widehat{f}_{\lambda}, \widehat{h}\}, \{\widehat{g}_{\mu}, \widehat{k}\} \in \Gamma$ with $\widehat{f}_{\lambda} \in \widehat{\mathfrak{N}}_{\lambda}(T)$ and $\widehat{g}_{\mu} \in \widehat{\mathfrak{N}}_{\mu}(T)$ one has

$$(3.6) \quad (\lambda - \bar{\mu})(\widehat{f}_{\lambda}, \widehat{g}_{\mu})_{\mathfrak{H}} = (h', k)_{\mathcal{H}} - (h, k')_{\mathcal{H}}, \quad \lambda, \mu \in \mathbb{C} \setminus \mathbb{R},$$

which follows from the identity (3.1). Hence, the subspace $\widehat{\mathfrak{N}}_{\lambda}(T)$ is positive in the Krein space $(\mathfrak{H}^2, J_{\mathfrak{H}})$ for $\lambda \in \mathbb{C}_+$ and negative for $\lambda \in \mathbb{C}_-$.

Corollary 3.4. *Let $\Gamma : \mathfrak{H}^2 \mapsto \mathcal{H}^2$ be a boundary relation for a symmetric operator A . Then:*

- (i) $n_{\pm}(A) \leq \dim \mathcal{H}$;
- (ii) if $n_{\pm}(A) < \infty$, then $\dim \mathcal{H} - n_{\pm}(A) = \dim \text{mul } \Gamma$;
- (iii) if $\dim \mathcal{H} < \infty$, then $n_+(A) = n_-(A)$.

Proof. (i) Let $\widetilde{A} = \mathcal{J}(\Gamma)$ be the main transform of Γ . It follows from [15, Lemma 2.14] that

$$n_{\pm}(A) = n_{\pm}(\widetilde{S}),$$

where $\widetilde{S} = \text{mul } \Gamma \subset \mathcal{H}^2$; cf. (3.3). This implies the statement (i).

(ii) If $n_{\pm}(A) < \infty$, or equivalently, $n_{\pm}(\widetilde{S}) < \infty$, one obtains

$$\dim \text{mul } \Gamma = \dim \widetilde{S} = \dim \mathcal{H} - \dim n_{\pm}(\widetilde{S}),$$

where the last identity follows from the fact that $\dim \widetilde{S}$ in \mathcal{H}^2 is equal to $\dim \text{ran}(\widetilde{S} - \lambda)$ in \mathcal{H} for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

(iii) If $\dim \mathcal{H} < \infty$, then clearly $n_+(\widetilde{S}) = n_-(\widetilde{S})$ and therefore also $n_+(A) = n_-(A)$. \square

Recall from [15] that the boundary relation $\Gamma : \mathfrak{H}^2 \mapsto \mathcal{H}^2$ is said to be *minimal*, if

$$\mathfrak{H} = \mathfrak{H}_{min} := \overline{\text{span}} \{ \mathfrak{N}_\lambda(T) : \lambda \in \mathbb{C}_+ \cup \mathbb{C}_- \}.$$

Since $\mathfrak{N}_\lambda(T)$ is dense in $\mathfrak{N}_\lambda(S^*)$, the boundary relation $\Gamma : \mathfrak{H}^2 \mapsto \mathcal{H}^2$ is minimal if and only if S is simple.

Definition 3.5. The *Weyl family* $M(\cdot)$ of S corresponding to the boundary relation $\Gamma : \mathfrak{H}^2 \mapsto \mathcal{H}^2$ is defined by $M(\lambda) := \Gamma(\widehat{\mathfrak{N}}_\lambda(T))$, i.e.,

$$(3.7) \quad M(\lambda) := \left\{ \widehat{h} \in \mathcal{H}^2 : \{ \widehat{f}_\lambda, \widehat{h} \} \in \Gamma \text{ for some } \widehat{f}_\lambda = \{ f, \lambda f \} \in \mathfrak{H}^2 \right\},$$

where $\lambda \in \mathbb{C} \setminus \mathbb{R}$. In the case where $M(\cdot)$ is operator-valued it is called the *Weyl function* of S corresponding to the boundary relation Γ .

Definition 3.6. The γ -field $\gamma(\cdot)$ of S corresponding to the boundary relation $\Gamma : \mathfrak{H}^2 \rightarrow \mathcal{H}^2$ is defined by

$$(3.8) \quad \gamma(\lambda) := \left\{ \{ h, f_\lambda \} \in \mathcal{H} \times \mathfrak{H} : \{ \widehat{f}_\lambda, \widehat{h} \} \in \Gamma \text{ for some } \widehat{f}_\lambda = \{ f, \lambda f \} \in \mathfrak{H}^2 \right\},$$

where $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Moreover, $\widehat{\gamma}(\lambda)$ stands for

$$(3.9) \quad \widehat{\gamma}(\lambda) := \left\{ \{ h, \widehat{f}_\lambda \} \in \mathcal{H} \times \mathfrak{H}^2 : \{ h, f_\lambda \} \in \gamma(\lambda), \quad \widehat{f}_\lambda = \{ f, \lambda f \} \in \mathfrak{H}^2 \right\}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

Associate with Γ the following linear relations which are not necessarily closed:

$$(3.10) \quad \begin{aligned} \Gamma_0 &= \left\{ \{ \widehat{f}, h \} : \{ \widehat{f}, \widehat{h} \} \in \Gamma, \widehat{h} = \{ h, h' \} \right\}, \\ \Gamma_1 &= \left\{ \{ \widehat{f}, h' \} : \{ \widehat{f}, \widehat{h} \} \in \Gamma, \widehat{h} = \{ h, h' \} \right\}. \end{aligned}$$

It is clear that

$$\text{dom } M(\lambda) = \Gamma_0(\widehat{\mathfrak{N}}_\lambda(T)) \subset \text{ran } \Gamma_0, \quad \text{ran } M(\lambda) = \Gamma_1(\widehat{\mathfrak{N}}_\lambda(T)) \subset \text{ran } \Gamma_1.$$

If the boundary relation Γ is single-valued the triplet $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ will be called a *boundary triplet* associated with the boundary relation $\Gamma : \mathfrak{H}^2 \mapsto \mathcal{H}^2$. In this case the Weyl family corresponding to the boundary triplet $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ can be also defined via the equality

$$(3.11) \quad \Gamma_1(\{ f_\lambda, \lambda f_\lambda \}) = M(\lambda) \Gamma_0(\{ f_\lambda, \lambda f_\lambda \}), \quad \{ f_\lambda, \lambda f_\lambda \} \in T.$$

The γ -field $\gamma(\cdot)$ associated with the boundary relation $\Gamma : \mathfrak{H}^2 \rightarrow \mathcal{H}^2$ is the first component of the mapping $\widehat{\gamma}(\lambda)$ in (3.9). Observe that

$$\widehat{\gamma}(\lambda) := (\Gamma_0 \upharpoonright \widehat{\mathfrak{N}}_\lambda(T))^{-1}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R},$$

is a linear mapping from $\Gamma_0(\widehat{\mathfrak{N}}_\lambda(T)) = \text{dom } M(\lambda)$ onto $\widehat{\mathfrak{N}}_\lambda(T)$; it is single-valued in view of (3.6). Consequently, the γ -field is a single-valued mapping from $\text{dom } M(\lambda)$ onto $\mathfrak{N}_\lambda(T)$ and satisfies $\gamma(\lambda) \Gamma_0 \widehat{f}_\lambda = f_\lambda$ for all $\widehat{f}_\lambda \in \widehat{\mathfrak{N}}_\lambda(T)$.

3.2. Realization theorem. It follows from the identity (3.6) that each Weyl family is a Nevanlinna family. In [15] the converse statement was also proven: each Nevanlinna family can be realized as the Weyl family of a minimal boundary relation.

Two boundary relations $\Gamma^{(j)} : (\mathfrak{H}^{(j)})^2 \rightarrow \mathcal{H}^2$, $j = 1, 2$, are said to be *unitarily equivalent* if there is a unitary operator $U : \mathfrak{H}^{(1)} \rightarrow \mathfrak{H}^{(2)}$ such that

$$(3.12) \quad \Gamma^{(2)} = \left\{ \left\{ \begin{pmatrix} Uf \\ Uf' \end{pmatrix}, \begin{pmatrix} h \\ h' \end{pmatrix} \right\} : \left\{ \begin{pmatrix} f \\ f' \end{pmatrix}, \begin{pmatrix} h \\ h' \end{pmatrix} \right\} \in \Gamma^{(1)} \right\}.$$

If the boundary relations $\Gamma^{(1)}$ and $\Gamma^{(2)}$ satisfy (3.12) and $S_j = \ker \Gamma^{(j)}$, $T_j = \text{dom } \Gamma^{(j)}$, $j = 1, 2$, then $S_2 = US_1U^{-1}$ and $T_2 = UT_1U^{-1}$.

Theorem 3.7. [15] *Let $\Gamma : \mathfrak{H}^2 \rightarrow \mathcal{H}^2$ be a boundary relation for S^* . Then the corresponding Weyl family $M(\cdot)$ belongs to the class $\tilde{R}(\mathcal{H})$.*

Conversely, if $M(\cdot)$ belongs to the class $\tilde{R}(\mathcal{H})$ then there exists a unique (up to unitary equivalence) minimal boundary relation whose Weyl function coincides with $M(\cdot)$.

In the following proposition geometric characterization of boundary relations, whose Weyl functions belong to certain subclasses of $\tilde{R}(\mathcal{H})$ is given.

Proposition 3.8. [15] *Let $\Gamma : \mathfrak{H}^2 \rightarrow \mathcal{H}^2$ be a boundary relation for S^* with the Weyl family $M(\lambda) = \Gamma(\widehat{\mathfrak{N}}_\lambda(T))$. Then:*

- (i) $M(\cdot) \in R(\mathcal{H})$ if and only if $\text{mul } \Gamma \cap (\{0\} \times \mathcal{H}) = \{0\}$;
- (ii) $M(\cdot) \in R^s(\mathcal{H})$ if and only if $\text{ran } \Gamma$ is dense in \mathcal{H}^2 ;
- (iii) $M(\cdot) \in R[\mathcal{H}]$ if and only if $\Gamma_0(\widehat{\mathfrak{N}}_\lambda(T)) = \mathcal{H}$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$;
- (iv) $M(\cdot) \in R^s[\mathcal{H}]$ if and only if $\text{mul } \Gamma_0 = \{0\}$ and $\Gamma_0(\widehat{\mathfrak{N}}_\lambda(T)) = \mathcal{H}$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$.
- (v) $M(\cdot) \in R^u[\mathcal{H}]$ if and only if $\text{ran } \Gamma = \mathcal{H}^2$.

The case (ii) is specified in more detail in the following Proposition.

Proposition 3.9. *Let $\Gamma : \mathfrak{H}^2 \rightarrow \mathcal{H}^2$ be a boundary relation for S^* and let $M(\cdot) = \{\Phi(\cdot), \Psi(\cdot)\}$ be the corresponding Weyl family. Then*

$$(3.13) \quad \dim \text{mul } \Gamma = \dim \ker \mathsf{N}_{\Phi, \Psi}.$$

In particular,

$$\text{mul } \Gamma = \{0\} \Leftrightarrow \ker \mathsf{N}_{\Phi, \Psi} = \{0\}.$$

Proof. Let $T(\lambda_0)$ be a mapping from \mathcal{H} to \mathcal{H}^2 given by $T(\lambda) = \begin{pmatrix} \Phi(\lambda) \\ \Psi(\lambda) \end{pmatrix}$. Then $M(\lambda_0) = T(\lambda_0)\mathcal{H}$. If $\mathcal{H}_0 := \ker \mathsf{N}_{\Phi, \Psi}(\lambda_0, \lambda_0) \neq 0$, then $T(\lambda_0)\mathcal{H}_0$ is the isotropic subspace of the space $T(\lambda_0)\mathcal{H}$ considered as a subspace of the Krein space $(\mathcal{H}^2, J_{\mathcal{H}})$. Therefore,

$$T(\lambda)\ker \mathsf{N}_{\Phi, \Psi}(\lambda, \lambda) = M(\lambda) \cap M(\lambda)^*.$$

In view of [15, Lemma 4.1] this yields the equality (3.9). \square

3.3. Linear transformations of boundary relations. Let $\Gamma : \mathfrak{H}^2 \rightarrow \mathcal{H}^2$ be a boundary relation for S^* and let W be a linear relation from the Krein space $(\mathcal{H}^2, J_{\mathcal{H}})$ to the Krein space $(\mathcal{K}^2, J_{\mathcal{K}})$. When the product WT is a boundary relation for S^* , then the corresponding γ -field can be expressed in terms of the γ -field of the original boundary relation. The Weyl family for WT can be expressed as a Shmul'yan transform of the original Weyl family.

Proposition 3.10. *Let $\Gamma : \mathfrak{H}^2 \rightarrow \mathcal{H}^2$ be a boundary relation for S^* with the γ -field $\gamma(\lambda)$ and the Weyl family $M(\lambda)$. Let W be a linear relation from the Kreĭn space $(\mathcal{H}^2, J_{\mathcal{H}})$ to the Kreĭn space $(\mathcal{K}^2, J_{\mathcal{K}})$, such that*

$$(3.14) \quad W\Gamma \text{ is unitary, } \ker W = \{0\}.$$

Then:

- (i) *the relation $W\Gamma : \mathfrak{H}^2 \rightarrow \mathcal{K}^2$ is a boundary relation for S^* ;*
- (ii) *the γ -field $\gamma_W(\lambda)$ associated with $W\Gamma$ is given by*

$$(3.15) \quad \gamma_W(\lambda) = \{ \{k, f_{\lambda}\} \in \mathcal{K} \times \mathfrak{H} : \widehat{k} = W\widehat{h}, \{f_{\lambda}, \widehat{h}\} \in \Gamma \}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R};$$

- (iii) *the corresponding Weyl family $M_W(\lambda)$ is given by Shmul'yan transform*

$$(3.16) \quad M_W(\lambda) = W[M(\lambda)], \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

Proof. (i) This statement is immediate from Proposition 3.2, since $\ker W\Gamma = \ker \Gamma = S$.

(ii) According to (3.8) the γ -field associated to $W\Gamma$ is given by

$$\gamma_W(\lambda) = \{ \{k, f_{\lambda}\} \in \mathcal{K} \times \mathfrak{H} : \{f_{\lambda}, \widehat{k}\} \in W\Gamma \} = \{ \{k, f_{\lambda}\} \in \mathcal{K} \times \mathfrak{H} : \widehat{k} = W\widehat{h}, \{f_{\lambda}, \widehat{h}\} \in \Gamma \},$$

$\lambda \in \mathbb{C} \setminus \mathbb{R}$, which gives (3.15).

(iii) By Definition 3.5 the Weyl family $M_W(\cdot)$ of S corresponding to the boundary relation $\Gamma_W : \mathfrak{H}^2 \mapsto \mathcal{K}^2$ is given by

$$M_W(\lambda) = \left\{ \widehat{k} \in \mathcal{K}^2 : \{f_{\lambda}, \widehat{k}\} \in W\Gamma \right\} = \left\{ \widehat{k} \in \mathcal{K}^2 : \widehat{k} = W\widehat{h}, \{f_{\lambda}, \widehat{h}\} \in \Gamma \right\},$$

$\lambda \in \mathbb{C} \setminus \mathbb{R}$, and this leads to (3.16). \square

In order to guarantee that the product $W\Gamma$ in (3.14) is unitary, some sufficient conditions on W are required when Γ is not a standard unitary operator; cf. Section 2.

Remark 3.11. If W is a standard unitary operator from \mathcal{H}^2 to \mathcal{K}^2 , then the conditions in (3.14) are automatically satisfied, and the γ -field and the Weyl function of the boundary relation $W\Gamma$ can be written as

$$(3.17) \quad \gamma_W(\lambda) = \{ \{W_{00}h + W_{01}h', \gamma(\lambda)h\} : \{h, h'\} \in M(\lambda) \}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R},$$

and

$$(3.18) \quad M_W(\lambda) = \{ \{W_{00}h + W_{01}h', W_{10}h + W_{11}h'\} : \{h, h'\} \in M(\lambda) \}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R},$$

where W is decomposed as in (2.32). When $W = J_{\mathcal{H}}$ the boundary relation $W\Gamma$ takes the form

$$(3.19) \quad \Gamma^{\top} := \Gamma_{J_{\mathcal{H}}} = \left\{ \left\{ \widehat{f}, J_{\mathcal{H}}\widehat{h} \right\} : \{\widehat{f}, \widehat{h}\} \in \Gamma \right\},$$

and is called the *transposed boundary relation*. As follows from (3.18) the corresponding Weyl family $M^{\top}(\cdot)$ for Γ^{\top} coincides with $-M(\cdot)^{-1}$.

Theorem 3.12. *Let W be a standard unitary operator in the Kreĭn space $(\mathcal{H}^2, J_{\mathcal{H}})$. The following classes of Nevanlinna families are invariant under the Shmul'yan transform induced by W :*

- (i) *the class $\widetilde{R}(\mathcal{H})$ of all Nevanlinna families;*
- (ii) *the class $\widetilde{R}^c(\mathcal{H})$ of constant Nevanlinna families;*
- (iii) *the class $R^s(\mathcal{H})$ of strict Nevanlinna functions;*
- (iv) *the class $R^u(\mathcal{H}) = R^u[\mathcal{H}]$ of uniformly strict Nevanlinna functions.*

Proof. (i) According to Theorem 3.7 every Nevanlinna family $M \in \widetilde{R}(\mathcal{H})$ admits a realization as a Weyl family of a boundary relation Γ . By Lemma 3.10 the linear fractional transform $\widetilde{M} = WM$ is the Weyl family of the boundary relation $W\Gamma$ and, therefore by Theorem 3.7, \widetilde{M} belongs to $\widetilde{R}(\mathcal{H})$.

(ii) Clearly, if $M(\lambda)$ does not depend on $\lambda \in \mathbb{C} \setminus \mathbb{R}$, the same is true for $\widetilde{M} = WM$.

(iii)&(iv) Since W is a unitary operator in the Krein space $(\mathcal{H}^2, J_{\mathcal{H}})$, W maps \mathcal{H}^2 onto \mathcal{H}^2 and, furthermore, as a bounded everywhere defined unitary operator it also maps a dense subspace of \mathcal{H} onto a dense subspace of \mathcal{H} . Therefore, (iii) and (iv) follow from parts (v) and (ii) of Proposition 3.8, respectively. \square

The invariance property for the class $R^u(\mathcal{H}) = R^u[\mathcal{H}]$ in (iv) of Theorem 3.12 was proved in a completely different manner by M.G. Krein and Yu. L. Shmul'yan [36]; the present proof reflects the power of the realization in Theorem 3.7.

4. SPECIAL BOUNDARY RELATIONS AND THEIR WEYL FAMILIES

In this section special attention is paid to the boundary relations whose Weyl families belong to the class $R[\mathcal{H}]$. In particular, an orthogonal decomposition of the auxiliary space \mathcal{H} leads to Weyl functions of intermediate extensions. Furthermore, attention is paid to the subclasses $R^s[\mathcal{H}]$ and $R^u[\mathcal{H}]$ of strict and uniformly strict Nevanlinna functions.

4.1. Ordinary boundary triplets. ([25]) Let S be a closed symmetric operator in a Hilbert space \mathfrak{H} with equal defect numbers. A triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$, where \mathcal{H} is a Hilbert space with $\dim \mathcal{H} = n_{\pm}(S)$ and $\Gamma_i \in [S^*, \mathcal{H}]$, $i = 0, 1$, is said to be an *ordinary boundary triplet* (or a *boundary value space*) for S^* if:

(A1) the abstract Green's identity

$$(4.1) \quad (f', g) - (f, g') = (\Gamma_1 \widehat{f}, \Gamma_0 \widehat{g})_{\mathcal{H}} - (\Gamma_0 \widehat{f}, \Gamma_1 \widehat{g})_{\mathcal{H}}$$

holds for all $\widehat{f} = \{f, f'\}, \widehat{g} = \{g, g'\} \in S^*$;

(A2) the mapping $\Gamma := \{\Gamma_0, \Gamma_1\} : S^* \rightarrow \mathcal{H}^2$ is surjective.

For a densely defined symmetric operator this notion was introduced by A.N. Kochubei [30] (see also [25]), a close definition has been used for other purpose by A.V. Štraus [47]. For a nondensely defined symmetric operator it was introduced in [38]. In this case the adjoint S^* of a symmetric operator S in \mathfrak{H} is a closed linear relation in \mathfrak{H} ; it can be considered as a Hilbert space with the graph norm.

Simple observations (see [15, Proposition 5.3]) show that the following statement holds.

Proposition 4.1. *The following statements are equivalent:*

- (i) *a triplet $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is an ordinary boundary triplet for S^* ;*
- (ii) *$\Gamma = \{\Gamma_0, \Gamma_1\} : \mathfrak{H}^2 \mapsto \mathcal{H}^2$ is a boundary relation for S^* such that $\text{ran } \Gamma = \mathcal{H}^2$;*
- (iii) *the corresponding Weyl family $M(\cdot)$ belongs to $R^u[\mathcal{H}]$.*

A linear extension \widetilde{A} of the operator S is said to be *intermediate* if $S \subset \widetilde{A} \subset S^*$. Ordinary boundary triplets provide a means to describe all intermediate extensions of S . It is well-known (see [17, 38]) that the set of all intermediate extensions of A in \mathfrak{H} admits the parametrization

$$(4.2) \quad \widetilde{A}_{\Theta} := \{\widehat{f} \in A^* : \Gamma \widehat{f} \in \Theta\} = \ker(\Gamma_1 - \Theta \Gamma_0)$$

where Θ ranges over the set of all linear relations in \mathcal{H} . Moreover, in this case the linear relation \tilde{A}_Θ is closed (symmetric, selfadjoint) if and only if the linear relation Θ is closed (symmetric, selfadjoint, respectively).

The definitions of the Weyl function $M(\cdot)$ and the γ -field $\gamma(\cdot)$ corresponding to the ordinary boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ can be rewritten in a simpler form

$$(4.3) \quad \widehat{\gamma}(\lambda) := (\Gamma_0 \upharpoonright \widehat{\mathfrak{N}}_\lambda)^{-1}, \quad \gamma(\lambda) := \pi_1(\Gamma_0 \upharpoonright \widehat{\mathfrak{N}}_\lambda)^{-1}, \quad M(\lambda) = \Gamma_1 \widehat{\gamma}(\lambda),$$

with $\lambda \in \rho(A_0)$. Here $\widehat{\mathfrak{N}}_\lambda := \widehat{\mathfrak{N}}_\lambda(S^*)$ and π_1 stands for the projection onto the first component of $\mathcal{H} \oplus \mathcal{H}$. The Weyl function $M(\cdot)$ and the γ -field $\gamma(\cdot)$ satisfy the following identities:

$$(4.4) \quad \gamma(\lambda) = [I + (\lambda - \mu)(A_0 - \lambda)^{-1}] \gamma(\mu), \quad \lambda, \mu \in \mathbb{C}_+(\mathbb{C}_-).$$

$$(4.5) \quad M(\lambda) = M(\mu)^* + (\lambda - \bar{\mu}) \gamma(\mu)^* [I + (\lambda - \mu)(A_0 - \lambda)^{-1}] \gamma(\mu), \quad \lambda, \mu \in \mathbb{C} \setminus \mathbb{R},$$

In the case of an ordinary boundary triplet the resolvent of an intermediate extension \tilde{A} of A can be calculated in terms of the corresponding Weyl function.

Proposition 4.2. [19] *Let $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be an ordinary boundary triplet for S^* , let $M(\cdot)$ be the corresponding Weyl function, let Θ be a linear relation in \mathcal{H} , and let $\lambda \in \rho(A_0)$. Then $\lambda \in \rho(\tilde{A}_\Theta)$ if and only if $0 \in \rho(\Theta - M(\lambda))$ and the resolvent of \tilde{A}_Θ is given by*

$$(4.6) \quad (\tilde{A}_\Theta - \lambda)^{-1} = (A_0 - \lambda)^{-1} + \gamma(\lambda)(\Theta - M(\lambda))^{-1} \gamma(\bar{\lambda})^*.$$

The following result is well known. However, the very simple proof is here derived from the definition of boundary triplet.

Proposition 4.3. *Two ordinary boundary triplets $\Pi^{(j)} = \{\mathcal{H}, \Gamma_0^{(j)}, \Gamma_1^{(j)}\}$ ($j = 1, 2$) for A^* are connected via the formula*

$$(4.7) \quad \Gamma^{(2)} = W \Gamma^{(1)}, \quad W = \begin{pmatrix} W_{00} & W_{01} \\ W_{10} & W_{11} \end{pmatrix},$$

where W is a $J_{\mathcal{H}}$ -unitary operator in $[\mathcal{H}^2]$.

Proof. By Proposition 4.1 ordinary boundary triplets determine single-valued unitary relations $\Gamma^{(j)} : \mathfrak{H}^2 \rightarrow \mathcal{H}^2$ satisfying $\text{dom } \Gamma^{(j)} = A^*$ and $\text{ran } \Gamma^{(j)} = \mathcal{H}^2$. The composition $W := \Gamma^{(2)} \circ \Gamma^{(1)}{}^{-1}$ is a bounded unitary mapping from \mathcal{H}^2 onto \mathcal{H}^2 such that $\Gamma^{(2)} = W \Gamma^{(1)}$ (see Theorem 2.13). The statement is now immediate from Lemma 3.10. \square

4.2. Ordinary boundary triplets and boundary relations. Let S be a closed symmetric relation in \mathfrak{H} with equal defect numbers. It turns out that all the boundary relations of S can be obtained by extending Proposition 4.3 in an appropriate manner. Namely, it is shown that they naturally arise when the group of J -unitary operators in $[\mathcal{H}^2]$ is augmented by the class of all J -unitary relations W in \mathcal{H}^2 for which $\ker W = \{0\}$.

Proposition 4.4. *Let S be a closed symmetric relation in \mathfrak{H} with equal defect numbers, let $\Gamma^{(1)} = \{\mathcal{H}, \Gamma_0^{(1)}, \Gamma_1^{(1)}\}$ be an ordinary boundary triplet for S^* and let W be a unitary relation from $(\mathcal{H}^2, J_{\mathcal{H}})$ to $(\mathcal{H}^2, J_{\tilde{\mathcal{H}}})$ such that $\ker W = \{0\}$. Then the linear relation*

$$(4.8) \quad \Gamma^{(2)} = W \Gamma^{(1)}$$

is a boundary relation for S^* .

Conversely, for every boundary relation $\Gamma^{(2)}$ of S^* there exists a unitary relation W with $\ker W = \{0\}$ and such that $\Gamma^{(2)}$ can be recovered from $\Gamma^{(1)}$ via (4.8).

In particular, the formula (4.8) establishes a bijective correspondence between the set of all boundary relations for S^* with the fixed parameter space \mathcal{H} and the set of all unitary relations W in $(\mathcal{H}^2, J_{\mathcal{H}})$ with $\ker W = \{0\}$.

Proof. Let W be a unitary relation from $(\mathcal{H}^2, J_{\mathcal{H}})$ to $(\tilde{\mathcal{H}}^2, J_{\tilde{\mathcal{H}}})$ such that $\ker W = \{0\}$ and let $\Gamma^{(2)}$ be given by (4.8). Then $\ker \Gamma^{(2)} = \ker \Gamma^{(1)} = S$. Moreover, since $\text{dom } \Gamma^{(1)} = S^*$ is closed and $\text{dom } W \subset \text{ran } \Gamma^{(1)} = \mathcal{H}^2$, part (v) of Theorem 2.13 shows that $\Gamma^{(2)}$ is a unitary relation from $(\mathfrak{H}^2, J_{\mathfrak{H}})$ to $(\tilde{\mathcal{H}}^2, J_{\tilde{\mathcal{H}}})$. Since $\ker \Gamma^{(2)} = S$, $\Gamma^{(2)}$ is a boundary relation for S^* .

Conversely, let $\Gamma^{(2)} : (\mathfrak{H}^2, J_{\mathfrak{H}}) \rightarrow (\tilde{\mathcal{H}}^2, J_{\tilde{\mathcal{H}}})$ be a boundary relation for S^* . Then by (iv) of Theorem 2.13 the linear relation $W^{-1} := \Gamma^{(1)} \circ (\Gamma^{(2)})^{-1}$ is a unitary relation from $(\tilde{\mathcal{H}}^2, J_{\tilde{\mathcal{H}}})$ to $(\mathcal{H}^2, J_{\mathcal{H}})$, since $\text{dom } \Gamma^{(1)} = S^*$ is closed and $\Gamma^{(2)}$ as a boundary relation for S^* satisfies

$$\text{ran } (\Gamma^{(2)})^{-1} = \text{dom } \Gamma^{(2)} \subset S^* = \text{dom } \Gamma^{(1)}.$$

Assume that $h \in \ker W$. Then $\{h, 0\} \in W = \Gamma^2 \circ (\Gamma^{(1)})^{-1}$ and, hence, there is a vector $g \in \mathfrak{H}^2$ such that

$$\{h, g\} \in (\Gamma^{(1)})^{-1}, \quad \{g, 0\} \in \Gamma^{(2)}.$$

Since $\ker \Gamma^{(2)} = S$, this implies that $g \in S$ and

$$\{g, h\} \in \Gamma^{(1)}.$$

Since $\Gamma^{(1)} = \{\mathcal{H}, \Gamma_0^{(1)}, \Gamma_1^{(1)}\}$ is an ordinary boundary triplet for S^* this implies $h = 0$. This shows that $\ker W = \{0\}$, and completes the proof of the converse statement. \square

Remark 4.5. (i) A relation $\Gamma^{(2)}$ from $(\mathfrak{H}^2, J_{\mathfrak{H}})$ to $(\mathcal{H}^2, J_{\mathcal{H}})$ is a single-valued boundary relation for S^* if and only if W in (4.7) is a unitary operator in the parameter space $(\mathcal{H}^2, J_{\mathcal{H}})$ with $\ker W = \{0\}$.

(ii) If $\dim \mathcal{H} < \infty$ then $\text{dom } W$ and, therefore, also $\text{ran } W$ is closed, and one has

$$\dim(\text{dom } W) + \dim(\ker W) = \dim \mathcal{H}^2, \quad \dim(\text{ran } W) + \dim(\text{mul } W) = \dim \tilde{\mathcal{H}}^2.$$

If W is single-valued then the assumption $\ker W = \{0\}$ is equivalent to the fact that W is a standard unitary operator in $[\mathcal{H}^2, \tilde{\mathcal{H}}^2]$, in which case $\dim \tilde{\mathcal{H}} = \dim \mathcal{H}$; cf. Corollary 2.11. If instead W is a unitary relation from \mathcal{H}^2 to $\tilde{\mathcal{H}}^2$ with $\ker W = \{0\}$, then $\dim \mathcal{H} \leq \dim \tilde{\mathcal{H}}$ and

$$\dim \tilde{\mathcal{H}} - \dim \mathcal{H} = \dim \text{mul } W.$$

(iii) If $\mathcal{H}_1 = \mathcal{H}_2 =: \mathcal{H}$, then $\dim \ker W = \dim \text{mul } W$. Therefore if $\ker W \neq \{0\}$, then $\text{mul } W \neq \{0\}$ and $\Gamma^{(2)}$ is a multi-valued mapping. In this case $\Gamma^{(2)}$ is a boundary relation for S_1^* , where $S_1 := \ker \Gamma^{(2)} \supset A$, and $S_1 \neq A$.

Corollary 4.6. Let S be a closed symmetric relation in \mathfrak{H} with equal defect numbers (n, n) , $n \leq \infty$, and let $\Gamma^{(1)} : \mathfrak{H}^2 \rightarrow \mathcal{H}^2$ be an ordinary boundary triplet for S^* . Then the class of all single-valued boundary relations $\Gamma^{(2)} : \mathfrak{H}^2 \rightarrow \mathcal{H}^2$ for S^* satisfying $\text{dom } \Gamma^{(2)} = S^*$ coincides with the class of ordinary boundary triplets for S^* ; they are parametrized by the class of all standard unitary operators $W \in [\mathcal{H}^2]$ via (4.8).

Furthermore, if $n = \dim \mathcal{H} < \infty$ then the class of all boundary relations for S^* with the fixed parameter space \mathcal{H} coincides with the class of ordinary boundary triplets for S^* .

Proof. Let $\Gamma^{(2)} : \mathfrak{H}^2 \rightarrow \mathcal{H}^2$ be a single-valued boundary relation for S^* with $\text{dom } \Gamma^{(2)} = S^*$. Then $\text{ran } \Gamma^{(2)}$ is closed and dense in \mathcal{H}^2 , so that in (4.8) $\mathcal{H}^2 = \text{ran } \Gamma^{(2)} \subset \text{ran } W$, and consequently W is a standard unitary operator in \mathcal{H}^2 . Therefore, $\Gamma^{(2)}$ is an ordinary boundary triplet for S^* .

If the defect numbers of S are finite, then in (4.8) $\dim \text{mul } \Gamma^{(2)} = \dim \text{mul } W = 0$ by Remark 4.5. Therefore, $\Gamma^{(2)}$ is single-valued and since $n < \infty$, $\text{dom } \Gamma^{(2)}$ is closed, which means that $\text{dom } \Gamma^{(2)} = S^*$. Therefore, by the first part of the proof $\Gamma^{(2)}$ is an ordinary boundary triplet for S^* . \square

The next result gives a complete description of all the Weyl families of a symmetric relation S with equal finite defect numbers in an arbitrary parameter space \mathcal{K} .

Proposition 4.7. *Let S be a closed symmetric relation in \mathfrak{H} with equal defect numbers (n, n) , $n \leq \infty$, let $\Gamma^{(1)} : \mathfrak{H}^2 \rightarrow \mathcal{H}^2$ be an ordinary boundary triplet for S^* with the Weyl family $M(\lambda)$, and let $\Gamma^{(2)} = W\Gamma^{(1)} : \mathfrak{H}^2 \rightarrow \mathcal{H}^2$ be an arbitrary boundary relation for S^* . Then the Weyl family associated with $\Gamma^{(2)}$ is the Shmul'yan transform of $M(\lambda)$ under W : $M^{(2)}(\lambda) = W[M(\lambda)] =: M_W(\lambda)$.*

Furthermore, the class of all Weyl families of boundary relations $\tilde{\Gamma}^{(2)} : \mathfrak{H}^2 \rightarrow \mathcal{K}^2$ of S^ in the parameter space \mathcal{K} with $\dim \mathcal{K} = \dim \mathcal{H}$ are unitarily equivalent to the class of all Weyl families $M_W(\lambda)$ of S acting on \mathcal{H} , and they are connected to each others by unitarily equivalent Shmul'yan transforms.*

If $\dim \mathcal{K} > \dim \mathcal{H}$ (so that $\dim \mathcal{H} < \infty$), then the strict part $M_r(\lambda)$ of $M(\lambda)$ in \mathcal{K} is unitarily equivalent to a Weyl function of S in the parameter space \mathcal{H} .

Proof. The first assertion $M^{(2)}(\lambda) = W[M(\lambda)]$ is clear from Proposition 4.4 and the definition of the Weyl family.

Next consider boundary relations $\tilde{\Gamma}^{(2)} : \mathfrak{H}^2 \rightarrow \mathcal{K}^2$ of S^* in the parameter space \mathcal{K} with $\dim \mathcal{K} = \dim \mathcal{H}$. Let U be a unitary mapping from the Hilbert space \mathcal{H} onto the Hilbert space \mathcal{K} and define the unitary mapping \tilde{U} from $\mathcal{H} \oplus \mathcal{H}$ onto $\mathcal{K} \oplus \mathcal{K}$ by $\tilde{U} = U \oplus U$. Then \tilde{U} is also a standard unitary mapping from $(\mathcal{H}^2, J_{\mathcal{H}})$ onto $(\mathcal{K}^2, J_{\mathcal{K}})$ in the Krein space sense. Hence $\tilde{\Gamma}^{(2)}$ is represented in the form $\tilde{\Gamma}^{(2)} = \tilde{U}\Gamma^{(2)}$, where $\Gamma^{(2)} : \mathfrak{H}^2 \rightarrow \mathcal{H}^2$ is a boundary relation for S^* and clearly all boundary relations for S^* with the parameter space \mathcal{K} are obtained in this way. The corresponding Weyl families are connected by $\tilde{M}^{(2)}(\lambda) = \tilde{U}M^{(2)}(\lambda)$, and hence they are unitarily equivalent. Moreover, if $\tilde{\Gamma}^{(1)} = \tilde{U}\Gamma^{(1)}$ and one defines $\tilde{W} = \tilde{U}W\tilde{U}^{-1}$, where W is as in Proposition 4.4, then \tilde{W} is a unitary relation in $(\mathcal{K}^2, J_{\mathcal{K}})$ with $\ker \tilde{W} = \{0\}$. Moreover, the corresponding Shmul'yan transforms are unitarily equivalent: $\tilde{W}[\tilde{M}^{(2)}(\lambda)] = \tilde{U}W[M^{(2)}(\lambda)]$.

If $\dim \mathcal{K} > \dim \mathcal{H}$ with $n = \dim \mathcal{H} < \infty$ then S has finite defect numbers. Consequently, $\text{dom } \tilde{\Gamma}^{(2)} = S^*$, $\text{ran } \tilde{\Gamma}^{(2)}$ is closed, and for \tilde{W} as in Proposition 4.4 one has $\text{dom } \tilde{W} = \mathcal{H}^2 = M(\lambda) \dot{+} M(\lambda)^*$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, since $\Gamma^{(1)}$ is an ordinary boundary triplet for S^* . The codimension of $\text{mul } \tilde{\Gamma}^{(2)}$ is $2n$ in $\text{ran } \tilde{\Gamma}^{(2)}$ and n in $\tilde{\Gamma}^{(2)}(\widehat{\mathfrak{N}}_{\lambda}(S^*))$ for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Since

$$\tilde{M}^{(2)}(\lambda) \cap (\{0\} \oplus \mathcal{K}) = \{0\} \oplus \text{mul } \tilde{M}^{(2)}(\lambda) = \text{mul } \tilde{\Gamma}^{(2)},$$

one concludes that the codimension of $\text{mul } \tilde{\Gamma}_0^{(2)}$ in $\text{dom } \tilde{M}^{(2)}(\lambda)$ is also n ; see [15, Lemma 4.1]. Moreover, here $\text{mul } \tilde{\Gamma}_0^{(2)} = \ker(\tilde{M}^{(2)}(\lambda) - \tilde{M}^{(2)}(\lambda)^*)$ and therefore also the codimension of $\overline{\text{mul } \tilde{\Gamma}_0^{(2)}}$ in $\overline{\text{dom } \tilde{M}^{(2)}(\lambda)}$ is n for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Since $\overline{\text{dom } \tilde{M}^{(2)}(\lambda)} = \overline{\text{ran } \tilde{\Gamma}_0^{(2)}}$ by [15, Corollary 4.3], it follows that $\overline{\text{ran } \tilde{\Gamma}_0^{(2)}} = \overline{\text{dom } \tilde{M}^{(2)}(\lambda)}$ for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Thus $\tilde{M}^{(2)}(\lambda) \in \tilde{R}_{inv}(\mathcal{K})$

and “the strict part” $\widetilde{M}_r^{(2)}(\lambda)$ of $\widetilde{M}^{(2)}(\lambda)$ acts on an n -dimensional subspace of \mathcal{K} . It is obvious that $\widetilde{M}_r^{(2)}(\lambda)$ defines a Weyl function for S^* on an n -dimensional parameter space and, hence, it is unitarily equivalent to a Weyl function of S acting on \mathcal{H} . \square

Proposition 4.7 shows that for studying the Weyl families of arbitrary boundary relations for a given symmetric relation S it is enough to select one parameter space \mathcal{H} whose dimension is equal to n , the defect numbers of S . The strict part of the Weyl family is the one that determines the symmetric operator and its selfadjoint extension in the model space up to unitary equivalence. However, for instance, in the connection of generalized resolvents nonstrict Weyl families naturally appear.

4.3. Boundary relations whose Weyl functions belong to the class $R[\mathcal{H}]$. A purely geometric characterization of this class of boundary relations is given in the following proposition.

Proposition 4.8. *Let S be a closed symmetric relation in a Hilbert space \mathfrak{H} . Let \mathcal{H} be a Hilbert space and let $\Gamma : \mathfrak{H}^2 \rightarrow \mathcal{H}^2$ be a (possibly multivalued) linear relation such that:*

- (B1) *Green’s identity (3.1) holds;*
- (B2) *$\text{ran } \Gamma_0 = \mathcal{H}$;*
- (B3) *$A_0 := \ker \Gamma_0$ is a selfadjoint linear relation in \mathfrak{H} .*

Then $\Gamma : \mathfrak{H}^2 \rightarrow \mathcal{H}^2$ is a boundary relation for $S^ := (\ker \Gamma)^*$ such that*

$$(4.9) \quad \Gamma_0(\widehat{\mathfrak{N}}_\lambda(T)) = \mathcal{H}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

Conversely, every closed isometric linear relation $\Gamma : \mathfrak{H}^2 \rightarrow \mathcal{H}^2$ satisfying (4.9) satisfies also the conditions (B1)–(B3).

If the conditions (B1)–(B3) are satisfied, the corresponding Weyl function belongs to the class $R[\mathcal{H}]$. Moreover, every $R[\mathcal{H}]$ -function is the Weyl function of some boundary relation $\Gamma : \mathfrak{H}^2 \rightarrow \mathcal{H}^2$ with the properties (B1)–(B3).

Proof. The proof of the direct statement was given in [15].

Assume now that $\Gamma : \mathfrak{H}^2 \rightarrow \mathcal{H}^2$ is a closed isometric linear relation satisfying (4.9). Let $\{\widehat{f}_\lambda, \widehat{h}\}, \{\widehat{g}_\lambda, \widehat{k}\} \in \Gamma$ with

$$\widehat{f}_\lambda = \begin{pmatrix} f_\lambda \\ \lambda f_\lambda \end{pmatrix} \in \widehat{\mathfrak{N}}_\lambda(T), \quad \widehat{g}_\lambda = \begin{pmatrix} g_\lambda \\ \bar{\lambda} g_\lambda \end{pmatrix} \in \widehat{\mathfrak{N}}_{\bar{\lambda}}(T), \quad \widehat{h} = \begin{pmatrix} h \\ h' \end{pmatrix}, \quad \widehat{k} = \begin{pmatrix} k \\ k' \end{pmatrix} \in \mathcal{H}^2.$$

Then it follows from (3.6) that

$$0 = (\lambda f_\lambda, g_\lambda)_{\mathfrak{H}} - (f_\lambda, \bar{\lambda} g_\lambda)_{\mathfrak{H}} = (h', k)_{\mathcal{H}} - (h, k')_{\mathcal{H}}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

Since $\widehat{h} \in M(\lambda)$, $\widehat{k} \in M(\bar{\lambda})$ this implies that $M(\lambda) \subset M(\bar{\lambda})^*$. Next, the assumption (4.9), implies that

$$\text{dom } M(\lambda) = \text{dom } M(\bar{\lambda}) = \mathcal{H}.$$

and, hence, $M(\lambda)$ is bounded for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Since the operator $M(\lambda)$ is dissipative for $\lambda \in \mathbb{C}_+$ this implies

$$\text{ran } (\Gamma \widehat{\mathfrak{N}}_\lambda + \lambda) = \text{ran } (M(\lambda) + \lambda) = \mathcal{H}, \quad \lambda \in \mathbb{C}_+.$$

Due to Proposition 3.3 this proves that $\Gamma : \mathfrak{H}^2 \rightarrow \mathcal{H}^2$ is a boundary relation for $S = T^*$.

Thus $\mathcal{H} = \Gamma_0(\widehat{\mathfrak{N}}_\lambda(T)) \subset \text{ran } \Gamma_0$, so that $\text{ran } \Gamma_0 = \mathcal{H}$, i.e., (B2) is satisfied. Also the property (B3) is obtained from $\Gamma_0(\widehat{\mathfrak{N}}_\lambda(T)) = \mathcal{H}$ by using [15, Proposition 4.15]. The condition (B1) for the boundary relation Γ is clearly satisfied.

The fact that every $R[\mathcal{H}]$ -function is the Weyl function of some boundary relation $\Gamma : \mathfrak{H}^2 \rightarrow \mathcal{H}^2$ satisfying the conditions (B1)-(B3) is implied by Theorem 3.7 and Proposition 3.8. \square

Recall that for a boundary relation $\Gamma : \mathfrak{H}^2 \rightarrow \mathcal{H}^2$ satisfying the conditions (B1)-(B3) the operator function $\gamma(\lambda) = \pi_1(\Gamma_0 \upharpoonright \widehat{\mathfrak{N}}_\lambda(T))^{-1} : \mathcal{H} \rightarrow \mathfrak{N}_\lambda(T)$ is bounded and single-valued for every $\lambda \in \mathbb{C} \setminus \mathbb{R}$, see [15]. Clearly, the Weyl function $M(\cdot)$ and the γ -field $\gamma(\cdot)$ satisfy the identities (4.4) and (4.5). Let $E(t)$ be the spectral family of A_0 and let $P = E(\infty)$ be the orthogonal projection onto $\overline{\text{dom}} A_0$. Then (4.5) leads to the following integral representation of $M(\lambda)$

$$(4.10) \quad (M(\lambda)h, h) = a_h + b_h\lambda + \int_{\mathbb{R}} \left(\frac{1}{t - \lambda} - \frac{t}{t^2 + 1} \right) d\sigma_h(t), \quad h \in \mathcal{H}_0,$$

where

$$a_h = (\text{Re } M(i)h, h)_{\mathcal{H}}, \quad b_h = ((I - P)\gamma(i)h, \gamma(i)h), \quad d\sigma_h(t) = (t^2 + 1)d(E(t)P\gamma(i)h, P\gamma(i)h)_{\mathfrak{H}}.$$

The representation (4.10) leads to the following characterization.

Proposition 4.9. *Let S be a symmetric operator in \mathfrak{H} . Let $\Gamma : \mathfrak{H}^2 \rightarrow \mathcal{H}^2$ be a boundary relation for S^* satisfying the conditions (B1)-(B3) and let $M(\lambda)$ be the corresponding Weyl function. Let $\mathcal{H}_0 = \pi_1 \text{mul } \Gamma$, $A_0 = \ker \Gamma_0$, and $T = \text{dom } \Gamma$. Then:*

(i) $\text{mul } A_0 = \{0\}$ if and only if

$$(4.11) \quad \lim_{y \rightarrow \infty} \frac{(M(iy)h, h)_{\mathcal{H}}}{iy} = 0, \quad h \in \mathcal{H};$$

(ii) $\text{mul } T = \{0\}$ if and only if M satisfies the condition (4.11) and

$$(4.12) \quad \lim_{y \uparrow \infty} y \text{Im} (M(iy)h, h) = \infty, \quad h \in \mathcal{H} \ominus \mathcal{H}_0.$$

Proof. The first statement is immediate from the equality

$$(4.13) \quad \lim_{y \rightarrow \infty} \frac{(M_s(iy)h, h)_{\mathcal{H}}}{iy} = \|(I - P)\gamma(i)h\|_{\mathfrak{H}}^2 = \|(I - P)\gamma(\mu)h\|_{\mathfrak{H}}^2.$$

Under the assumption (4.11) the limit in (4.12) takes the form

$$\lim_{y \uparrow \infty} y \text{Im} (M(iy)h, h) = \int_{\mathbb{R}} (t^2 + 1) d\|E_t \gamma(i)h\|_{\mathfrak{H}}^2.$$

Remark that the mapping $\gamma(i)$ restricted to $\mathcal{H} \ominus \mathcal{H}_0$ is injective and, hence, this limit is finite for some $h \in \mathcal{H} \ominus \mathcal{H}_0$, $h \neq 0$, if and only if $\mathfrak{N}_i(T) \cap \text{dom } A_0 = (A_0 - \lambda)^{-1}(\text{mul } T)$ is nontrivial. For the proof of the last equality see [15]. \square

The boundary relations with the additional properties (B1)–(B3) are invariant under a special class of transforms, cf. Proposition 3.10. Let $B \in [\mathcal{H}]$ and let $G \in [\mathcal{H}]$ be invertible, and assume that

$$BG = (BG)^*.$$

Define the block operator \widetilde{W} by

$$(4.14) \quad \widetilde{W} = \begin{pmatrix} G^{-1} & 0 \\ B & G^* \end{pmatrix}, \quad \text{with } BG = (BG)^*.$$

It is easy to see that \widetilde{W} is a $J_{\mathcal{H}}$ -unitary operator in \mathcal{H}^2 .

Proposition 4.10. *Let $\Gamma : \mathfrak{H}^2 \rightarrow \mathcal{H}^2$ be a boundary relation for S^* which satisfies the conditions (B1)–(B3), let $\gamma(\lambda)$ and $M(\lambda)$ be the corresponding γ -field and the Weyl function, and moreover let $\widetilde{W} \in [\mathcal{H} \oplus \mathcal{H}]$ be given by (4.14). Then:*

(i) *the transform $\widetilde{\Gamma} = \widetilde{W}\Gamma$ of Γ given by*

$$(4.15) \quad \widetilde{\Gamma} = \left\{ \left\{ \widehat{f}, \begin{pmatrix} G^{-1}h \\ Bh + G^*h' \end{pmatrix} \right\} : \left\{ \widehat{f}, \widehat{h} \right\} \in \Gamma \right\},$$

is a boundary linear relation for S^ with $\text{dom } \Gamma_W = \text{dom } \Gamma$ and $\ker \Gamma_W = \ker \Gamma = S$ which also satisfies the conditions (B1)–(B3);*

(ii) *the γ -field and the Weyl function associated to $\widetilde{\Gamma}$ are given by*

$$(4.16) \quad \widetilde{\gamma}(\lambda) = \gamma(\lambda)G, \quad \widetilde{M}(\lambda) = BG + G^*M(\lambda)G \quad (\in [\mathcal{H}]), \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

Proof. (i) Since \widetilde{W} defined by (4.14) is a $J_{\mathcal{H}}$ -unitary operator in \mathcal{H} , the transform $\widetilde{\Gamma} = \widetilde{W}\Gamma : \mathfrak{H}^* \rightarrow \mathcal{H}^2$ is a boundary relation for S^* with $\text{dom } \Gamma_W = \text{dom } \Gamma$ and $\ker \Gamma_W = \ker \Gamma = S$ and clearly $\widetilde{\Gamma}$ admits the representation (4.15). Moreover, since $\text{ran } \Gamma_0 = \mathcal{H}$ and $G \in [\mathcal{H}]$ is invertible, the equality $\text{ran } \widetilde{\Gamma}_0 = \mathcal{H}$ holds and $\ker \widetilde{\Gamma}_0 = \ker \Gamma_0$ is selfadjoint. Hence, $\widetilde{\Gamma}$ satisfies the conditions (B1)–(B3).

(ii) By Lemma 3.10 the Weyl function $\widetilde{M}(\lambda)$ associated to $\widetilde{\Gamma}$ is given by

$$(4.17) \quad \begin{aligned} \widetilde{M}(\lambda) &= \{ \{G^{-1}h, Bh + G^*h'\} : \{h, h'\} \in M(\lambda) \} \\ &= \{ \{k, BGk + G^*M(\lambda)Gk\} : h = Gk \in \text{dom } M(\lambda) = \mathcal{H} \} \\ &= BG + G^*M(\lambda)G, \end{aligned}$$

where $BG = (BG)^*$. Similarly, the γ -field $\widetilde{\gamma}(\lambda)$ corresponding to $\widetilde{\Gamma}$ takes the form

$$(4.18) \quad \begin{aligned} \widetilde{\gamma}(\lambda) &= \{ \{G^{-1}h, \gamma(\lambda)h\} : \{h, h'\} \in M(\lambda) \} = \{ \{k, \gamma(\lambda)Gk\} : h = Gk \in \text{dom } M(\lambda) = \mathcal{H} \}, \\ \text{so that } \widetilde{\gamma}(\lambda) &= \gamma(\lambda)G, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}. \end{aligned} \quad \square$$

Remark 4.11. In the case when the transposed boundary relation Γ^{\top} satisfies (B1)–(B3) the corresponding Weyl family $M^{\top}(\cdot) = -M(\cdot)^{-1}$ is single-valued and belongs to the class $R[\mathcal{H}]$.

Up to this point boundary relations $\Gamma : (\mathfrak{H}^2, J_{\mathfrak{H}}) \rightarrow (\mathcal{H}^2, J_{\mathcal{H}})$ satisfying the conditions (B1)–(B3) are in general multi-valued. Next we briefly discuss the case when it is single-valued.

Definition 4.12. [19] If a boundary relation $\Gamma : (\mathfrak{H}^2, J_{\mathfrak{H}}) \rightarrow (\mathcal{H}^2, J_{\mathcal{H}})$ is single-valued and satisfies the conditions (B1)–(B3), then the triplet $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is said to be a *generalized boundary triplet*.

The following corollary is implied by Proposition 3.8.

Corollary 4.13. [19] A single-valued boundary relation $\Gamma = \{\Gamma_0, \Gamma_1\} : \mathfrak{H}^2 \mapsto \mathcal{H}^2$ corresponds to a generalized boundary triplet $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ if and only if the corresponding Weyl function $M(\cdot)$ belongs to the class $R^s[\mathcal{H}]$.

In the case of a generalized boundary triplet the last condition in Proposition 4.9 is simplified in the following way

$$(4.19) \quad \lim_{y \uparrow \infty} y \text{Im} (M(iy)h, h) = \infty, \quad h \in \mathcal{H}.$$

The next proposition shows how one can reduce a multi-valued boundary relation χ with the properties (B1)–(B3) to a single-valued boundary relation with the same properties (B1)–(B3).

Proposition 4.14. *Let $\Gamma : (\mathfrak{H}^2, J_{\mathfrak{H}}) \rightarrow (\mathcal{H}^2, J_{\mathcal{H}})$ be a multi-valued boundary relation which satisfies (B1)–(B3). Then:*

- (1) $\mathcal{H}_0 = \pi_0 \text{mul } \Gamma$ is a closed subspace of \mathcal{H} ($\mathcal{H}_1 = \mathcal{H} \ominus \mathcal{H}_0$);
- (2) $\text{mul } \Gamma$ is the graph of bounded symmetric operator $K_0 \in [\mathcal{H}_0, \mathcal{H}]$;
- (3) for every bounded selfadjoint extension K of K_0 in \mathcal{H} the linear relation

$$(4.20) \quad \Gamma' := \left\{ \left\{ \begin{pmatrix} f \\ f' \end{pmatrix}, \begin{pmatrix} P_{\mathcal{H}_1} h \\ h' - Kh \end{pmatrix} \right\} : \left\{ \begin{pmatrix} f \\ f' \end{pmatrix}, \begin{pmatrix} h \\ h' \end{pmatrix} \right\} \in \Gamma \right\} : \mathfrak{H}^2 \rightarrow \mathcal{H}_1^2$$

is a single-valued boundary relation satisfying (B1)–(B3). The Weyl functions $M(\lambda)$ and $M_1(\lambda)$, corresponding to the boundary relations Γ and Γ' are connected by

$$(4.21) \quad M(\lambda) = K + \text{diag}(0_{\mathcal{H}_0}, M_1(\lambda)), \quad (\lambda \in \mathbb{C}_+).$$

Proof. 1) Since $\text{ran } \Gamma_0 = \mathcal{H}$ one obtains from [15, Lemma 2.1] that $\text{ran } \Gamma \widehat{+} (\{0\} \oplus \mathcal{H})$ is closed. By [29, Theorem 4.8] and Proposition 2.7

$$(4.22) \quad \text{mul } \Gamma \widehat{+} (\{0\} \oplus \mathcal{H}) \text{ is closed.}$$

Using [15, Lemma 2.1] again one obtains $\mathcal{H}_0 := \pi_1 \text{mul } \Gamma$ is a closed subspace of \mathcal{H} .

2) It follows from (4.22) that $\text{mul } \Gamma$ is the graph of a bounded operator $K_0 : \mathcal{H}_0 \rightarrow \mathcal{H}$. Since $\text{mul } \Gamma$ is a neutral subspace in $(\mathfrak{H}^2, J_{\mathfrak{H}})$ the operator K_0 is symmetric in \mathcal{H} .

3) Let K be a bounded selfadjoint operator extension of K_0 , $K \in [\mathcal{H}]$. Since $\text{mul } \Gamma = \text{ran } \Gamma^{[\perp]}$, one obtains from

$$0 = (h', h_0) - (h, K_0 h_0) = (h' - Kh, h_0) \quad (h_0 \in \mathcal{H}_0), \{h, h'\} \in \text{ran } \Gamma,$$

that $h' - Kh$ is orthogonal to \mathcal{H}_0 . This proves that $\text{ran } \Gamma' \subset \mathcal{H}_1^2$. The mapping Γ' is single-valued since for $\{h, h'\} \in \text{mul } \Gamma$ one has

$$P_{\mathcal{H}_1} h = 0, \quad h' - Kh = K_0 h - Kh = 0.$$

Clearly, $\text{ran } \Gamma'_0 = \mathcal{H}_1$ since $\text{ran } \Gamma_0 = \mathcal{H}$. Assume that $\{f, f'\} \in \ker \Gamma'_0$. It means that there is a vector $h' \in \mathcal{H}$ such that $\{h, h'\} \in \text{mul } \Gamma$

$$(4.23) \quad \left\{ \begin{pmatrix} f \\ f' \end{pmatrix}, \begin{pmatrix} h \\ h' \end{pmatrix} \right\} \in \Gamma, \quad P_{\mathcal{H}_1} h = 0.$$

Then $h \in \mathcal{H}_0$ and, hence, there is a vector $h'' \in \mathcal{H}$ such that $\{h, h''\} \in \text{mul } \Gamma$. Therefore,

$$\left\{ \begin{pmatrix} f \\ f' \end{pmatrix}, \begin{pmatrix} 0 \\ h' - h'' \end{pmatrix} \right\} \in \Gamma,$$

and, hence, $\{f, f'\} \in \ker \Gamma_0$. This proves that Γ' satisfies (B3). The equality (4.21) is implied by (4.20). \square

Corollary 4.15. *If a boundary relation $\Gamma : (\mathfrak{H}^2, J_{\mathfrak{H}}) \rightarrow (\mathcal{H}^2, J_{\mathcal{H}})$ for S^* satisfies (B1)–(B3), then $n_+(S) = n_-(S)$.*

5. WEYL FUNCTIONS FOR INTERMEDIATE EXTENSIONS

Let S be a closed symmetric operator in a separable Hilbert space \mathfrak{H} and let $\Gamma : \mathfrak{H}^2 \rightarrow \mathcal{H}^2$ be a boundary relation for S^* which satisfies the conditions (B1)–(B3), so that the corresponding Weyl family $M(\lambda)$ belongs to the class $R[\mathcal{H}]$. The purpose of this section is to associate intermediate symmetric extensions H of S to different types of Nevanlinna functions (say, linear combinations of M_{ij} , Schur complements and compressions of linear fractional transformations of $M(\lambda)$), which are obtained as block transforms of the operator matrix representation of $M(\lambda)$ in

$$(5.1) \quad \mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2, \quad M(\lambda) = (M_{ij}(\lambda))_{i,j=1}^2.$$

Consider the linear relations

$$(5.2) \quad \mathcal{P}^{(j)} = \left\{ \left\{ \begin{pmatrix} h \\ h' \end{pmatrix}, \begin{pmatrix} h \\ P_{\mathcal{H}_j} h' \end{pmatrix} \right\} : h \in \mathcal{H}_j, h' \in \mathcal{H} \right\}, \quad j = 1, 2,$$

which, clearly, are unitary from $(\mathcal{H}^2, J_{\mathcal{H}})$ to $(\mathcal{H}_j^2, J_{\mathcal{H}_j})$. In general, it is not clear whether $\mathcal{P}^{(j)} \circ \Gamma$ is a unitary relation if $\text{ran } \Gamma \not\subseteq \text{dom } \mathcal{P}^{(j)} = \mathcal{H}_j \times \mathcal{H}$ (cf. Theorem 2.13). However, in the case when $\Gamma : \mathfrak{H}^2 \rightarrow \mathcal{H}^2$ satisfies the conditions (B1)–(B3) it turns out that $\mathcal{P}^{(j)} \circ \Gamma$ is a unitary relation from $(\mathcal{H}^2, J_{\mathcal{H}})$ to $(\mathcal{H}_j^2, J_{\mathcal{H}_j})$, $(j = 1, 2)$.

Proposition 5.1. *Let $\Gamma : \mathfrak{H}^2 \rightarrow \mathcal{H}^2$ be a boundary relation for S^* which satisfies the conditions (B1)–(B3), let $\gamma(\lambda)$ be the corresponding γ -field and decompose the corresponding Weyl function $M(\lambda)$ as in (5.1). Then:*

(i) *the linear relation H_1 given by*

$$(5.3) \quad H^{(1)} = \left\{ \widehat{f} \in S^* : \left\{ \widehat{f}, \begin{pmatrix} 0 \\ h' \end{pmatrix} \right\} \in \Gamma \text{ for some } h' \in \mathcal{H}_2 \right\},$$

is closed and symmetric in \mathfrak{H} and has equal defect numbers;

(ii) *the linear relation $\Gamma^{(1)} : \mathfrak{H}^2 \rightarrow \mathcal{H}_1^2$ given by*

$$(5.4) \quad \Gamma^{(1)} := \mathcal{P}^{(1)} \circ \Gamma = \left\{ \left\{ \widehat{f}, \begin{pmatrix} h \\ P_{\mathcal{H}_j} h' \end{pmatrix} \right\} : \left\{ \widehat{f}, \begin{pmatrix} h \\ h' \end{pmatrix} \right\} \in \Gamma \text{ for some } h \in \mathcal{H}_1, h' \in \mathcal{H} \right\},$$

is a boundary relation for $(H^{(1)})^$ which satisfies the conditions (B1)–(B3);*

(iii) *the domain $T_1 := \text{dom } \Gamma^{(1)}$ is dense in H_1^* and it can be rewritten as*

$$(5.5) \quad T^{(1)} = \left\{ \widehat{f} \in S^* : \{ \widehat{f}, \widehat{h} \} \in \Gamma, \pi_2 h = 0 \right\};$$

(iv) *the corresponding γ -field $\gamma_1(\lambda) : \mathcal{H}_1 \rightarrow \mathfrak{H}$ and the Weyl function $M_1(\lambda) \in [\mathcal{H}_1]$ are given by*

$$(5.6) \quad \gamma_1(\lambda) = \gamma(\lambda) \upharpoonright \mathcal{H}_1, \quad M_1(\lambda) = M_{11}(\lambda), \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

Proof. (i) By definition $\Gamma^{(1)}$ is a multi-valued mapping from \mathfrak{H}^2 into \mathcal{H}_1^2 . It satisfies the Green's identity (3.1), since for all $\{ \widehat{f}, \widehat{h} \}, \{ \widehat{g}, \widehat{k} \} \in \Gamma$ with $h, k \in \mathcal{H}_1$ one has

$$(f', g)_{\mathfrak{H}} - (f, g')_{\mathfrak{H}} = (h', k)_{\mathcal{H}} - (h, k')_{\mathcal{H}} = (\pi_1 h', k)_{\mathcal{H}} - (h, \pi_1 k')_{\mathcal{H}}.$$

The property (B2) of Γ implies that $\text{ran } \Gamma_0^{(1)} = \mathcal{H}_1$. Moreover, from the property (B3) of Γ one concludes that $\ker \Gamma_0^{(1)} = \ker \Gamma_0 = A_0$ is selfadjoint. Hence, $\Gamma^{(1)}$ is a boundary relation for $(H^{(1)})^*$ which admits the properties (B1)–(B3).

(ii) Since $\Gamma^{(1)}$ is unitary, $H^{(1)} = \ker \Gamma^{(1)}$ is closed and symmetric. The description of $H^{(1)}$ in (5.3) is immediate from the definition of $\Gamma^{(1)}$ in (ii). Since $\Gamma^{(1)}$ satisfies the conditions (B1)–(B3) the defect numbers of $H^{(1)}$ are equal to (n_1, n_1) , $n_1 = \dim \mathcal{H}_1 - \dim \ker \Gamma^{(1)}$.

(iii) The description of $T_1 = \text{dom } \Gamma^{(1)}$ in (5.5) is clear from the definition of $\Gamma^{(1)}$ in (i) and the denseness of $T^{(1)}$ in $(H^{(1)})^*$, or equivalently, the identity $(T^{(1)})^* = \ker \Gamma^{(1)} = H^{(1)}$ holds by the definition of boundary relations.

(iv) According to [15, Proposition 5.9] the conditions (B1)–(B3) imply that

$$(5.7) \quad \Gamma_0(\widehat{\mathfrak{N}}_\lambda(T)) = \mathcal{H}, \quad \Gamma_0^{(1)}(\widehat{\mathfrak{N}}_\lambda(T^{(1)})) = \mathcal{H}_1, \quad \text{for all } \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

Hence, $\text{dom } \widehat{\gamma}(\lambda) = \mathcal{H}$ and $\text{dom } \widehat{\gamma}_1(\lambda) = \mathcal{H}_1$, and the formulas

$$\gamma(\lambda) = \{ \{h, \widehat{f}_\lambda\} : \{\widehat{f}_\lambda, \widehat{h}\} \in \Gamma \}, \quad \gamma_1(\lambda) = \{ \{h, \widehat{f}_\lambda\} : \{\widehat{f}_\lambda, \widehat{h}\} \in \Gamma, h \in \mathcal{H}_1 \}$$

show that these single-valued mappings are connected by $\gamma_1(\lambda) = \gamma(\lambda) \upharpoonright \mathcal{H}_1$. Moreover, (5.7) implies that $M_1(\lambda) \in [\mathcal{H}_1]$, $M(\lambda) \in [\mathcal{H}]$, and thus

$$\begin{aligned} M_1(\lambda) &= \{ \widehat{h} \in \mathcal{H}_1^2 : \{\widehat{f}_\lambda, \widehat{h}\} \in \Gamma^{(1)} \} \\ &= \{ \{h, \pi_1 h'\} \in \mathcal{H}^2 : \{\widehat{f}_\lambda, \widehat{h}\} \in \Gamma, h \in \mathcal{H}_1 \} \\ &= P_{\mathcal{H}_1} M(\lambda) \upharpoonright \mathcal{H}_1. \end{aligned}$$

This completes the proof. \square

Replacing $P_{\mathcal{H}_1}$ by $P_{\mathcal{H}_2}$ one obtains

Corollary 5.2. *Let $\Gamma : \mathfrak{H}^2 \rightarrow \mathcal{H}^2$, $\gamma(\lambda)$, and $M(\lambda)$ be as in Proposition 5.1. Then:*

(i) *the linear relation H_2 given by*

$$(5.8) \quad H^{(2)} = \left\{ \widehat{f} \in S^* : \left\{ \widehat{f}, \begin{pmatrix} 0 \\ h' \end{pmatrix} \right\} \in \Gamma \text{ for some } h' \in \mathcal{H}_1 \right\},$$

is closed and symmetric in \mathfrak{H} and has equal defect numbers;

(ii) *the linear relation $\Gamma^{(2)} : \mathfrak{H}^2 \rightarrow \mathcal{H}_2^2$ given by*

$$\Gamma^{(2)} := \mathcal{P}^{(2)} \circ \Gamma = \left\{ \left\{ \widehat{f}, \begin{pmatrix} h \\ \pi_2 h' \end{pmatrix} \right\} : \left\{ \widehat{f}, \begin{pmatrix} h \\ h' \end{pmatrix} \right\} \in \Gamma \text{ for some } h \in \mathcal{H}_2, h' \in \mathcal{H} \right\},$$

is a boundary relation for H_2^ which satisfies the conditions (B1)–(B3);*

(iii) *the domain $T^{(2)} := \text{dom } \Gamma^{(2)}$ is dense in H_2^* and it can be rewritten as*

$$(5.9) \quad T^{(2)} = \left\{ \widehat{f} \in S^* : \{\widehat{f}, \widehat{h}\} \in \Gamma, \pi_1 h = 0 \right\};$$

(iv) *the corresponding γ -field $\gamma_2(\lambda) : \mathcal{H}_2 \rightarrow \mathfrak{H}$ and the Weyl function $M_2(\lambda) \in [\mathcal{H}_2]$ are given by*

$$(5.10) \quad \gamma_2(\lambda) = \gamma(\lambda) \upharpoonright \mathcal{H}_2, \quad M_2(\lambda) = M_{22}(\lambda), \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

Corollary 5.3. *Let $\Gamma : \mathfrak{H}^2 \rightarrow \mathcal{H}^2$ be a boundary relation for S^* , such that Γ , Γ^\top , $(\Gamma^{(2)})^\top$ satisfy the conditions (B1)–(B3), and decompose the corresponding Weyl function $M(\lambda)$ as in (5.1).*

(i) *the linear relation $S^{(1)}$ given by*

$$(5.11) \quad S^{(1)} = \left\{ \widehat{f} \in S^* : \left\{ \widehat{f}, \begin{pmatrix} h \\ 0 \end{pmatrix} \right\} \in \Gamma \text{ for some } h \in \mathcal{H}_2 \right\},$$

is closed and symmetric in \mathfrak{H} and has equal defect numbers;

(ii) *the linear relation*

$$\Gamma' := (\mathcal{P}^{(1)} \circ \Gamma^\top)^\top = \left\{ \left\{ \widehat{f}, \begin{pmatrix} P_{\mathcal{H}_1} h \\ h' \end{pmatrix} \right\} : \left\{ \widehat{f}, \begin{pmatrix} h \\ h' \end{pmatrix} \right\} \in \Gamma \text{ for some } h \in \mathcal{H}, h' \in \mathcal{H}_1 \right\},$$

is a boundary relation for $(S^{(1)})^$ which satisfies the conditions (B1)–(B3);*

(iii) *the corresponding Weyl function $M^{(1)}(\lambda) \in [\mathcal{H}_1]$ is given by*

$$(5.12) \quad M^{(1)}(\lambda) = M_{11}(\lambda) - M_{12}(\lambda)M_{22}(\lambda)^{-1}M_{21}(\lambda), \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

Proof. It follows from the assumptions and Proposition 4.8 that both $M(\cdot)$ and $M(\cdot)^{-1}$ belong to the class $R[\mathcal{H}]$ and $M(\cdot)^{-1}$ admits a block representation $M(\cdot)^{-1} = ((M(\cdot)^{-1})_{ij})_{i,j=1}^2$. Since $(\Gamma^{(2)})^\top$ satisfy the conditions (B1)–(B3), one obtains from Proposition 4.8 that $M_{22}(\cdot)^{-1}$ belongs to the class $R[\mathcal{H}_2]$. Then it follows from the Frobenius formula that

$$(5.13) \quad (M(\lambda)^{-1})_{11} = (M_{11}(\lambda) - M_{12}(\lambda)M_{22}(\lambda)^{-1}M_{21}(\lambda))^{-1}.$$

Let us apply Proposition 5.1 to the linear relation Γ^\top . Then the linear relation $\mathcal{P}^{(1)} \circ \Gamma^\top$ satisfies the assumptions (B1)–(B3) and the corresponding Weyl function coincides with $(M(\cdot)^{-1})_{11}$ in (5.13). To complete the proof it remains to show that the linear relation $(\mathcal{P}^{(1)} \circ \Gamma^\top)^\top$ satisfies the assumptions (B1)–(B3). Since the Weyl function

$$M'(\lambda) = (M(\cdot)^{-1})_{11} = M_{11}(\lambda) - M_{12}(\lambda)M_{22}(\lambda)^{-1}M_{21}(\lambda)$$

belongs to the class $R[\mathcal{H}_1]$ this fact is implied by Proposition 4.8. However, we will present also a direct proof.

Let $h_1 \in \mathcal{H}_1$. Since Γ satisfies (B2) there exists $\widehat{f} \in S^*$, and $h' \in \mathcal{H}$ such that

$$(5.14) \quad \left\{ \widehat{f}, \begin{pmatrix} h_1 \\ h'_1 \end{pmatrix} \right\} \in \Gamma.$$

Next, using the fact that $(\Gamma^{(2)})^\top$ satisfies (B2), we find $\widehat{g} \in S^*$, and $h_2 \in \mathcal{H}_2$, $h'_2 \in \mathcal{H}$, such that

$$(5.15) \quad \left\{ \widehat{g}, \begin{pmatrix} h_2 \\ h'_2 \end{pmatrix} \right\} \in \Gamma, \quad P_{\mathcal{H}_2} h'_2 = P_{\mathcal{H}_2} h'_1.$$

Now it follows from (5.14), (5.15) that

$$\left\{ \widehat{f} - \widehat{g}, \begin{pmatrix} h_1 - h_2 \\ h'_1 - h'_2 \end{pmatrix} \right\} \in \Gamma, \quad P_{\mathcal{H}_2} (h'_1 - h'_2) = 0.$$

This implies that $\text{ran } \Gamma'_0 = \mathcal{H}_1$ and, hence, Γ' satisfies (B2). The assumption (B3) for Γ' is implied by the equality

$$\ker \Gamma'_0 = \left\{ \widehat{f} \in S^* : \left\{ \widehat{f}, \begin{pmatrix} h_1 \\ h'_1 \end{pmatrix} \right\} \in \Gamma \right\}.$$

□

Proposition 5.4. *Let $\Gamma : \mathfrak{H}^2 \rightarrow \mathcal{H}^2$ be a boundary relation for S^* which satisfies the conditions (B1)–(B3), let $\gamma(\lambda)$ be the corresponding γ -field, let $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, decompose the corresponding Weyl function $M(\lambda)$ as in (5.1), and let $T \in [\mathcal{H}_2, \mathcal{H}_1]$. Then:*

(i) the linear relation H_T defined by

$$(5.16) \quad H_T = \left\{ \widehat{f} \in S^* : \{\widehat{f}, \widehat{h}\} \in \Gamma, h = 0, h'_2 = -T^*h'_1 \right\},$$

is closed and symmetric in \mathfrak{H} and has equal defect numbers;

(ii) the linear relation $\Gamma_T : \mathfrak{H}^2 \rightarrow \mathcal{H}_2^2$ given by

$$(5.17) \quad \Gamma_T = \left\{ \left\{ \widehat{f}, \begin{pmatrix} h_2 \\ T^*h'_1 + h'_2 \end{pmatrix} \right\} : \{\widehat{f}, \widehat{h}\} \in \Gamma, h_1 = Th_2 \right\},$$

is a boundary relation for H_T^* which satisfies the conditions (B1)–(B3);

(iii) the domain of Γ_T is given by

$$(5.18) \quad \text{dom } \Gamma_T = \left\{ \widehat{f} \in S^* : \{\widehat{f}, \widehat{h}\} \in \Gamma, h_1 = Th_2 \right\};$$

(iv) the γ -field $\gamma_T(\lambda) : \mathcal{H}_2 \rightarrow \mathfrak{H}$ corresponding to the boundary relation Γ_T is given by

$$(5.19) \quad \gamma_T(\lambda) = \gamma_1(\lambda)T + \gamma_2(\lambda),$$

where $\gamma(\lambda) = (\gamma_1(\lambda) \ \ \gamma_2(\lambda)) : \mathcal{H}_1 \oplus \mathcal{H}_2 \rightarrow \mathfrak{H}$ is decomposed according to $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$;

(v) the Weyl function $M_T(\lambda)$ associated to Γ_T is of the form

$$(5.20) \quad M_T(\lambda) = T^*M_{11}(\lambda)T + T^*M_{12}(\lambda) + M_{21}(\lambda)T + M_{22}(\lambda).$$

Proof. Define the operator $G \in [\mathcal{H}]$, where $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, and the operator $W \in [\mathcal{H} \oplus \mathcal{H}]$ via the block formulas

$$G = \begin{pmatrix} I & T \\ 0 & I \end{pmatrix}, \quad W = \begin{pmatrix} G^{-1} & 0 \\ 0 & G^* \end{pmatrix},$$

respectively. Then G is invertible, $G^{-1} \in [\mathcal{H}]$, and W is $J_{\mathcal{H}}$ -unitary in $\mathcal{H}^2 = \mathcal{H} \oplus \mathcal{H}$. According to Proposition 4.10 the product $\widetilde{\Gamma} = WT : \mathfrak{H}^2 \rightarrow \mathcal{H}^2$ given by (4.15) is a J -unitary relation which satisfies the properties (B1)–(B3). Moreover, according to (4.16) the γ -field and the Weyl function associated to $\widetilde{\Gamma}$ are given by

$$(5.21) \quad \widetilde{\gamma}(\lambda)h = \gamma(\lambda)Gh = \gamma_1(\lambda)(h_1 + Th_2) + \gamma_2(\lambda)h_2,$$

and

$$(5.22) \quad \widetilde{M}(\lambda) = \begin{pmatrix} M_{11}(\lambda) & M_{11}(\lambda)T + M_{12}(\lambda) \\ T^*M_{11}(\lambda) + M_{21}(\lambda) & T^*M_{11}(\lambda)T + T^*M_{12}(\lambda) + M_{21}(\lambda)T + M_{22}(\lambda) \end{pmatrix},$$

respectively. Since

$$G^{-1}h = \begin{pmatrix} h_1 - Th_2 \\ h_2 \end{pmatrix}, \quad G^*h' = \begin{pmatrix} h'_1 \\ T^*h'_1 + h'_2 \end{pmatrix},$$

it follows from Corollary 5.2 that H_T in (5.16) is a closed symmetric relation in \mathfrak{H} and that $\Gamma_T : \mathfrak{H}^2 \rightarrow \mathcal{H}_2^2$ defined by (5.17) is a boundary relation for H_T^* which satisfies the conditions (B1)–(B3). Moreover, the formulas for the γ -field and the Weyl function in (5.19) and (5.20) are obtained by applying Corollary 5.2 to the formulas (5.21) and (5.22).

The formula (5.18) is immediate from the description of Γ_T in (5.17). \square

Corollary 5.5. *Let S_j be symmetric operators in Hilbert spaces \mathfrak{H}_j and let $\Gamma^{(j)} : \mathfrak{H}_j^2 \rightarrow \mathcal{H}^2$ be boundary relations for S_j^* which satisfy the conditions (B1)–(B3), and let $M_j(\lambda)$ be the corresponding Weyl functions of S_j , $j=1,2$. Then:*

(i) the linear relation
(5.23)
$$H^{(3)} = \left\{ \widehat{f} = \widehat{f}_1 \oplus \widehat{f}_2 : \left\{ \widehat{f}_1, \begin{pmatrix} 0 \\ h_1 \end{pmatrix} \right\} \in \Gamma^{(1)}, \left\{ \widehat{f}_2, \begin{pmatrix} 0 \\ -h_1 \end{pmatrix} \right\} \in \Gamma^{(2)} \text{ for some } h_1 \in \mathcal{H} \right\},$$

is closed and symmetric in \mathfrak{H} and has equal defect numbers;

(ii) the linear relation $\Gamma^{(3)} : \mathfrak{H}^2 \rightarrow \mathcal{H}^2$ given by
(5.24)
$$\Gamma^{(3)} := \left\{ \left\{ \widehat{f}_1 \oplus \widehat{f}_2, \begin{pmatrix} h \\ h_1 + h_2 \end{pmatrix} \right\} : \left\{ \widehat{f}_1, \begin{pmatrix} h \\ h_1 \end{pmatrix} \right\} \in \Gamma^{(1)}, \left\{ \widehat{f}_2, \begin{pmatrix} h \\ h_2 \end{pmatrix} \right\} \in \Gamma^{(2)} \text{ for some } h, h_1, h_2 \in \mathcal{H} \right\},$$

is a boundary relation for H^* which satisfies the conditions (B1)–(B3);

(iii) the corresponding Weyl function $M(\lambda)$ associated to $\Gamma^{(3)}$ is
(5.25)
$$M(\lambda) = M_1(\lambda) + M_2(\lambda), \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

Proof. To prove the statements (i)–(iii) it is enough to apply Proposition 5.4 to the boundary relation

$$\Gamma^{(1)} \oplus \Gamma^{(2)} := \left\{ \left\{ \widehat{f}_1 \oplus \widehat{f}_2, \widehat{h}_1 \oplus \widehat{h}_2 \right\} : \left\{ \widehat{f}_1, \widehat{h}_1 \right\} \in \Gamma^{(1)}, \left\{ \widehat{f}_2, \widehat{h}_2 \right\} \in \Gamma^{(2)} \text{ for some } h_1, h_2 \in \mathcal{H} \right\},$$

for $S_1^* \oplus S_2^*$ with the corresponding Weyl function $M(\lambda) = \text{diag}(M_1(\lambda), M_2(\lambda))$, setting there $T = I_{\mathcal{H}}$. \square

6. ORTHOGONAL COUPLINGS

6.1. Orthogonal coupling and boundary relations. Let \mathfrak{H}_1 and \mathfrak{H}_2 be arbitrary Hilbert spaces and let \tilde{A} be a selfadjoint linear relation in the orthogonal sum $\tilde{\mathfrak{H}} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$. Then the formula

$$(6.1) \quad S_j = \tilde{A} \cap \mathfrak{H}_j^2, \quad T_j = \left\{ \begin{pmatrix} P_j \varphi \\ P_j \varphi' \end{pmatrix} : \begin{pmatrix} \varphi \\ \varphi' \end{pmatrix} \in \tilde{A} \right\},$$

defines closed symmetric linear relations S_1 and S_2 , and not necessarily closed linear relations T_1 and T_2 , in \mathfrak{H}_1 and \mathfrak{H}_2 , respectively. The relation \tilde{A} can be interpreted as a selfadjoint extension of the orthogonal sum $S_1 \oplus S_2$. It is called the *orthogonal coupling* of S_1 and S_2 (or of T_1 and T_2), see [47]. The selfadjoint relation \tilde{A} is said to be *minimal* with respect to the Hilbert space \mathfrak{H}_j (j is fixed, $j=1,2$) if

$$(6.2) \quad \mathfrak{H}_1 \oplus \mathfrak{H}_2 = \overline{\text{span}} \left\{ \mathfrak{H}_j + (\tilde{A} - \lambda)^{-1} \mathfrak{H}_j : \lambda \in \rho(\tilde{A}) \right\}.$$

Associate with T_j the eigenspaces as in (2.2), (3.5),

$$(6.3) \quad \mathfrak{N}_\lambda(T_j) = \ker(T_j - \lambda), \quad \widehat{\mathfrak{N}}_\lambda(T_j) = \left\{ \begin{pmatrix} f \\ \lambda f \end{pmatrix} \in T_j : f \in \mathfrak{N}_\lambda(T_j) \right\}.$$

Observe that S_2 is connected to $\tilde{S} = \text{mul } \Gamma$ in (3.3) via $S_2 = -\tilde{S}$, cf. (2.20). Moreover, according to [15, Lemma 2.14] $\mathfrak{N}_\lambda(T_j)$ is dense in $\mathfrak{N}_\lambda(S_j^*)$ for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$, $j = 1, 2$.

Lemma 6.1. [15] *Let \tilde{A} be a selfadjoint linear relation in $\tilde{\mathfrak{H}} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$, and let the linear relations S_j and T_j , $j = 1, 2$, be defined by (6.1). Then:*

- (i) $\mathfrak{N}_\lambda(T_1) = P_1(\tilde{A} - \lambda)^{-1} \mathfrak{H}_2$, $\mathfrak{N}_\lambda(T_2) = P_2(\tilde{A} - \lambda)^{-1} \mathfrak{H}_1$;
- (ii) $\mathfrak{N}_\lambda(T_j)$ is dense in $\mathfrak{N}_\lambda(S_j^*)$ for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$, $j = 1, 2$;
- (iii) The defect numbers of S_1 and $-S_2$ coincide: $n_\pm(S_1) = n_\mp(S_2)$;

(iv) \tilde{A} is minimal with respect to \mathfrak{H}_1 (resp. \mathfrak{H}_2) if and only if S_2 (resp. S_1) is simple.

The main transform $\tilde{A} = \mathcal{J}(\Gamma)$ of a boundary relation Γ defined by (2.20) can be treated as an orthogonal coupling of symmetric operators A and $\tilde{S} = -\text{mul } \Gamma$. Then the first statement of the following proposition is just a reformulation of Proposition 2.17.

Proposition 6.2. *Let Γ be a subspace in $\mathfrak{H}^2 \times \mathcal{H}^2$ and let $S = \ker \Gamma$. Then Γ is a boundary relation for S^* if and only if $\tilde{A} = \mathcal{J}(\Gamma)$ is a selfadjoint relation in $\mathfrak{H} \oplus \mathcal{H}$. In this case the boundary relation Γ is minimal if and only if $\tilde{A} = \mathcal{J}(\Gamma)$ is a minimal selfadjoint extension of $-\tilde{S} = -\text{mul } \Gamma$.*

Proof. To prove the second statement let us mention first that the boundary relation $\Gamma : \mathfrak{H}^2 \rightarrow \mathcal{H}^2$ for S_2^* is minimal if and only if the symmetric linear relation S is simple, since $\mathfrak{N}_\lambda(T)$ are dense in $\mathfrak{N}_\lambda(S^*)$ (see Lemma 6.1, (ii)). Combining this with the statement (iv) of Lemma 6.1 one proves that Γ is minimal if and only if $\tilde{A} = \mathcal{J}(\Gamma)$ is a minimal selfadjoint extension of $-\tilde{S} = -\text{mul } \Gamma$. \square

6.2. Induced boundary relation. Let A be a symmetric operator in the Hilbert space \mathfrak{H}_1 and let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^* . Let \tilde{A} be a selfadjoint extension of A in the Hilbert space $\mathfrak{H}_1 \oplus \mathfrak{H}_2$ and define the linear relations S_2 and T_2 . There is a natural way to define a boundary relation for S_2 in the Hilbert space \mathfrak{H}_2 with corresponding Weyl family.

Theorem 6.3. *Let A be a symmetric operator in \mathfrak{H}_1 with equal defect numbers and let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be an ordinary boundary triplet for A^* . Then:*

(i) *If $\tilde{A} = \tilde{A}^*$ is a minimal selfadjoint exit space extension of A in $\tilde{\mathfrak{H}} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$ and S_2 is defined by (6.1), then the linear relation $\chi : \mathfrak{H}_2^2 \rightarrow \mathcal{H}^2$ defined by*

$$(6.4) \quad \chi = \left\{ \left\{ \widehat{f}_2, \begin{pmatrix} \Gamma_0 \widehat{f}_1 \\ -\Gamma_1 \widehat{f}_1 \end{pmatrix} \right\} : \widehat{f}_1 \oplus \widehat{f}_2 \in \tilde{A}, \widehat{f}_1 \in A^*, \widehat{f}_2 \in T_2 \right\}$$

is a minimal boundary relation for S_2^ .*

(ii) *If S_2 is a simple symmetric operator in \mathfrak{H}_2 and $\chi : \mathfrak{H}_2^2 \rightarrow \mathcal{H}^2$ is a minimal boundary relation for S_2^* , then the linear relation \tilde{A} defined by*

$$(6.5) \quad \tilde{A} = \left\{ \widehat{f}_1 \oplus \widehat{f}_2 \in A^* \oplus S_2^* : \left\{ \widehat{f}_2, \begin{pmatrix} \Gamma_0 \widehat{f}_1 \\ -\Gamma_1 \widehat{f}_1 \end{pmatrix} \right\} \in \chi \right\}$$

is a minimal selfadjoint extension of A which satisfies $\tilde{A} \cap \mathfrak{H}_2^2 = S_2$.

Proof. (i) Let \tilde{A} be a selfadjoint extension of A in the Hilbert space $\mathfrak{H}_1 \oplus \mathfrak{H}_2$ and define the linear relation

$$(6.6) \quad \Delta := \mathcal{J}^{-1} \circ \tilde{A} = \left\{ \left\{ \begin{pmatrix} f_1 \\ f'_1 \end{pmatrix}, \begin{pmatrix} f_2 \\ -f'_2 \end{pmatrix} \right\} : \{f_1 \oplus f_2, f'_1 \oplus f'_2\} \in \tilde{A}, f_j, f'_j \in \mathfrak{H}_j, j = 1, 2 \right\}.$$

It follows from Proposition 6.2 that Δ is a unitary relation from $(\mathfrak{H}_2^2, J_{\mathfrak{H}_2})$ to $(\mathfrak{H}_1^2, J_{\mathfrak{H}_1})$ with

$$\text{dom } \Delta = T_1, \quad \text{ran } \Delta = -T_2.$$

Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be an ordinary boundary triplet for A^* . Since

$$\text{ran } \Delta^{-1} = T_1 \subset \text{dom } \Gamma, \quad \text{ran } \Gamma = \mathcal{H}^2,$$

the composition χ_- of the unitary relations Δ^{-1} and Γ

$$(6.7) \quad \chi_- = \Gamma \circ \Delta^{-1} = \left\{ \left\{ \begin{pmatrix} f_2 \\ -f'_2 \end{pmatrix}, \Gamma \widehat{f}_1 \right\} : \widehat{f}_1 \oplus \widehat{f}_2 \in \widetilde{A}, \widehat{f}_1 \in A^*, \widehat{f}_2 \in T_2 \right\}.$$

is a unitary relation from $(\mathfrak{H}_2^2, J_{\mathfrak{H}_2})$ to $(\mathcal{H}^2, J_{\mathcal{H}})$ with $\text{dom } \chi_- = -T_2$, cf. Theorem 2.13. Changing signs in the second components of χ_- gives the linear relation χ of the form (6.4) with $\text{dom } \chi = T_2$. Since $\text{clos } T_2 = S_2^*$ it follows that $\chi : \mathfrak{H}_2^2 \rightarrow \mathcal{H}^2$ is a boundary relation for S_2^* .

To complete the proof of (i) it remains to prove that the boundary relation $\chi : \mathfrak{H}_2^2 \rightarrow \mathcal{H}^2$ for S_2^* is minimal. Since $\mathfrak{N}_\lambda(T_2)$ are dense in $\mathfrak{N}_\lambda(S_2^*)$ (see Lemma 6.1, (ii)) the latter is equivalent to simplicity of symmetric linear relation S_2 . But as was shown in Lemma 6.1 (iv), S_2 is simple if and only if \widetilde{A} is a minimal selfadjoint extension of A .

(ii) Let $\chi : \mathfrak{H}_2^2 \rightarrow \mathcal{H}^2$ be a boundary relation for S_2^* . Then

$$(6.8) \quad \chi_- = \left\{ \left\{ \begin{pmatrix} f_2 \\ -f'_2 \end{pmatrix}, \begin{pmatrix} h \\ -h' \end{pmatrix} \right\} : \left\{ \begin{pmatrix} f_2 \\ f'_2 \end{pmatrix}, \begin{pmatrix} h \\ h' \end{pmatrix} \right\} \in \chi \right\}.$$

is a boundary relation for $-S_2^*$. Since

$$\text{ran } \chi_- \subset \mathcal{H}^2 = \text{dom } \Gamma^{-1}, \quad \text{ran } \Gamma^{-1} = A^* = A^{[\perp]} = (\text{mul } \Gamma^{-1})^{[\perp]}$$

it follows from Theorem 2.13 that the linear relation

$$(6.9) \quad \Delta^{-1} := \Gamma^{-1} \circ \chi_- = \left\{ \left\{ \begin{pmatrix} f_2 \\ -f'_2 \end{pmatrix}, \widehat{f}_1 \right\} : \left\{ \begin{pmatrix} f_2 \\ f'_2 \end{pmatrix}, \begin{pmatrix} \Gamma_0 \widehat{f}_1 \\ -\Gamma_1 \widehat{f}_1 \end{pmatrix} \right\} \in \chi, \widehat{f}_1 \in A^* \right\}$$

is a unitary relation from $(\mathfrak{H}_2^2, J_{\mathfrak{H}_2})$ to $(\mathfrak{H}_1^2, J_{\mathfrak{H}_1})$. Therefore, the linear relation

$$(6.10) \quad \Delta = \left\{ \left\{ \widehat{f}_1, \begin{pmatrix} f_2 \\ -f'_2 \end{pmatrix} \right\} : \left\{ \begin{pmatrix} f_2 \\ f'_2 \end{pmatrix}, \begin{pmatrix} \Gamma_0 \widehat{f}_1 \\ -\Gamma_1 \widehat{f}_1 \end{pmatrix} \right\} \in \chi, \widehat{f}_1 \in A^* \right\}$$

is a unitary relation from $(\mathfrak{H}_1^2, J_{\mathfrak{H}_1})$ to $(\mathfrak{H}_2^2, J_{\mathfrak{H}_2})$. Due to Theorem 2.13 the composition $\Delta = \Gamma_-^{-1} \circ \chi$ of linear relations χ and Γ_-^{-1} is a unitary relation from $(\mathfrak{H}_2^2, J_{\mathfrak{H}_2})$ to $(\mathfrak{H}_1^2, J_{\mathfrak{H}_1})$. Applying the transform \mathcal{J} to the linear relation Δ one obtains by Proposition 6.2 a selfadjoint extension \widetilde{A} of A given by (6.5). Now (6.1) is implied by (6.5).

If $\chi : \mathfrak{H}_2^2 \rightarrow \mathcal{H}^2$ is a minimal boundary relation for S_2^* , then the minimality of \widetilde{A} with respect to \mathfrak{H}_1 is implied by the same reasons as in (i). \square

Proposition 6.4. *Let the assumptions of Theorem 6.3 be satisfied. Then the family $\tau(\lambda)$ defined by*

$$(6.11) \quad \tau(\lambda) = \left\{ \left\{ \Gamma_0 \widehat{f}_1, -\Gamma_1 \widehat{f}_1 \right\} : \widehat{f}_1 = P_{\mathfrak{H}_1} \widehat{f}, \widehat{f} \in \widetilde{A}, f' - \lambda f \in \mathfrak{H}_1 \right\},$$

is the Weyl family of S_2 corresponding to the boundary relation $\chi : \mathfrak{H}_2^2 \rightarrow \mathcal{H}^2$.

Proof. Let $\widehat{f}_2 = \{f_2, f'_2\} \in \mathfrak{N}_\lambda(T_2)$. Then it follows from (6.1) that there are vectors $f_1, f'_1 \in \mathfrak{H}_1$ such that $\widehat{f} = \{f, f'\} = \widehat{f}_1 \oplus \widehat{f}_2 \in \widetilde{A}$ and $f' - \lambda f \in \mathfrak{H}_1$. Hence by (6.5) one obtains

$$(6.12) \quad \left\{ \widehat{f}_2, \begin{pmatrix} \Gamma_0 \widehat{f}_1 \\ -\Gamma_1 \widehat{f}_1 \end{pmatrix} \right\} \in \chi.$$

Since $\widehat{f}_2 \in \widehat{\mathfrak{N}}_\lambda(T_2)$ this shows that $\{\Gamma_0\widehat{f}_1, -\Gamma_1\widehat{f}_1\}$ belongs to the Weyl family $M_\chi(\lambda)$ of S_2 corresponding to the boundary triplet χ . This proves the inclusion $M_\chi(\lambda) \subset \tau(\lambda)$, $\lambda \in \mathbb{C}_+ \cup \mathbb{C}_-$.

Conversely, if $\widehat{f}_1, \widehat{f}$ satisfy the conditions (6.11), then $\widehat{f}_2 = P_{\mathfrak{H}_2}\widehat{f}$ belongs to $\widehat{\mathfrak{N}}_\lambda(T_2)$. Due to (6.12) one obtains $\{\Gamma_0\widehat{f}_1, -\Gamma_1\widehat{f}_1\} \in M_\chi(\lambda)$, which proves the inclusion $\tau(\lambda) \subset M_\chi(\lambda)$. \square

Consider some examples of couplings of differential operators both single-valued and multi-valued.

Example 6.5. Let A be the symmetric differential operator in $L_2[0, 1]$ associated with the differential expression $-D^2$, whose domain of definition is given by

$$\text{dom } A = \{f \in C^1[0, 1] : f' \in AC[0, 1], f'' \in L_2[0, 1], f(0+) = f'(0+) = f(1) = 0\}.$$

Let S_2 be a symmetric differential operator $-D^2$ on the interval $[-1, 0]$, whose domain of definition is given by

$$\text{dom } S_2 = \{f \in C^1[-1, 0] : f' \in AC[-1, 0], f'' \in L_2[-1, 0], f(0-) = f'(0-) = f(-1) = 0\}.$$

Then the boundary conditions (6.5) take the form

$$f(0+) = f(0-), \quad f'(0+) = f'(0-),$$

and determine a selfadjoint operator \tilde{A} in $L_2[-1, 1]$ associated with the differential expression $-D^2$ and the boundary conditions

$$f(1) = 0, \quad f(-1) = 0.$$

Example 6.6. Let A be a minimal differential operator in $L_2[0, 1]$ associated with the differential expression $-D^2$. The domain of A is characterized by the following conditions

$$\text{dom } A = \{f \in C^1[0, 1] : f' \in AC[0, 1], f'' \in L_2[0, 1], f(0+) = f'(0+) = f(1) = f'(1) = 0\}.$$

Let the boundary triplet $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$ is given by

$$\Gamma_0 f = \begin{pmatrix} f(0+) \\ f(1) \end{pmatrix}, \quad \Gamma_1 f = \begin{pmatrix} f'(0+) \\ -f'(1) \end{pmatrix}.$$

Consider a minimal differential operator S_2 generated by the differential expression $-D^2$ on the interval $[-1, 0]$ and let the boundary triplet $\{\mathbb{C}, \chi_0, \chi_1\}$ for S_2^* is given by

$$\chi_0 f = \begin{pmatrix} f(0-) \\ f(-1) \end{pmatrix}, \quad \chi_1 f = \begin{pmatrix} -f'(0-) \\ f'(-1) \end{pmatrix}.$$

Then the boundary conditions (6.5) take the form

$$f(0+) = f(0-), \quad f'(0+) = f'(0-), \quad f(1) = f(-1), \quad f'(1) = f'(-1)$$

and determine a selfadjoint operator \tilde{A} in $L_2[-1, 1]$ associated with the differential expression $-D^2$ and the periodic boundary conditions

$$f(1) = f(-1), \quad f'(1) = f'(-1).$$

Theorem 6.3 establishes a one-to-one correspondence between all minimal with respect to \mathfrak{H}_1 exit space selfadjoint extensions of A and all minimal boundary relations $\chi : \mathfrak{H}_2^2 \rightarrow \mathcal{H}^2$ with a fixed space \mathcal{H} . Since minimal boundary relations are uniquely determined by their Weyl families, one can consider the correspondence established in Theorem 6.3 as a one-to-one correspondence between all minimal exit space selfadjoint extensions of A and all

Nevanlinna families $\tau(\cdot) \in \widetilde{R}(\mathcal{H})$. This correspondence can be written explicitly in terms of generalized resolvents (see Section 4).

Proposition 6.7. *Let under the assumptions of Proposition 6.3 $\tau = \{\Phi, \Psi\}$ be the Weyl family of the operator S_2 corresponding to the GBT (6.4). Then:*

$$(6.13) \quad \dim S_1/A = \dim \ker \mathbf{N}_{\Phi, \Psi}.$$

If, additionally, $S_1 = A$ then $T_1 \neq T_1^(= S_1^*)$ if and only if $\tau \in R_{\mathcal{H}}^s \setminus R_{\mathcal{H}}^u$, that is $0 \in \sigma_c(\mathbf{N}_{\Phi, \Psi}(\lambda, \lambda))$ for each $\lambda \in \mathbb{C}_+$.*

Proof. It follows from (6.4) that

$$\begin{pmatrix} h \\ h' \end{pmatrix} \in \text{mul } \chi \iff \begin{pmatrix} h \\ -h' \end{pmatrix} = \Gamma \widehat{f}, \text{ where } \widehat{f} \in \widetilde{A} \cap \mathfrak{H}_1^2 = S_1.$$

Since Γ is an isomorphism between linear spaces \mathcal{H}^2 and A^*/A this implies that $\text{mul } \chi$ and S_1/A are isomorphic. Therefore

$$\dim S_1/A = \dim \text{mul } \chi.$$

Making use of the equality (3.13) one obtains (6.13). \square

Example 6.8. Let A be the same as in the previous example and let S_2 be a minimal differential operator generated in $L_2(-\infty, 1)$ by the differential expression $-D^2$. Define a boundary relation $\chi : S_2^* \rightarrow \mathcal{H}^2$ ($\mathcal{H} = \mathbb{C}$) for S_2^* by the equality

$$\chi = \left\{ \left\{ \widehat{f}, \text{col} (f(0-), c, -f'(0-), hc) \right\} : \widehat{f} \in S_2^*, c \in \mathbb{C} \right\},$$

where $h \in \mathbb{R}$ is fixed. The equality (6.5) take the form

$$f(0+) = f(0-), \quad f'(0+) = f'(0-), \quad f(1) = c, \quad f'(1) = ch, \quad c \in \mathbb{C},$$

and determine a selfadjoint operator \widetilde{A} generated in $L_2(-\infty, 1]$ by the differential expression $-D^2$ and the boundary condition

$$f'(1) = hf(1).$$

The operator S_1 here is a restriction of $-D^2$ to the domain

$$\text{dom } S_1 = \{f \in C^1[0, 1] : f' \in AC[0, 1], f'' \in L_2[0, 1], f(0+) = f'(0+) = f'(1) - hf(1) = 0\},$$

and $\dim S_1/A = 1$.

The boundary relations $\chi^j : \mathfrak{H}_2^2 \rightarrow \mathcal{H}^2$ (see Theorem 6.3) are induced by the ordinary boundary triplets Π_j ($j = 1, 2$) (see (6.4)). Hence, due to Theorem 4.3 the connection between two Weyl families $\tau_j(\lambda)$ corresponding to boundary relations $\chi^j : \mathfrak{H}_2^2 \rightarrow \mathcal{H}^2$ can be explicitly expressed by means of the transform W which connects the Weyl functions $M_1(\lambda)$ and $M_2(\lambda)$.

Proposition 6.9. *Let the ordinary boundary triplets $\Pi_j = \{\mathcal{H}, \Gamma_0^j, \Gamma_1^j\}$ ($j = 1, 2$) for A^* be connected via the formula (4.7) and let $\chi^j : \mathfrak{H}_2^2 \rightarrow \mathcal{H}^2$ be boundary relations induced by the ordinary boundary triplets Π_j ($j = 1, 2$) via the formula (6.4). Then the boundary relations $\chi^j : \mathfrak{H}_2^2 \rightarrow \mathcal{H}^2$ and the corresponding Weyl families $\tau_j(\lambda)$ are connected by the formulas*

$$(6.14) \quad \chi^{(2)} = \widetilde{W} \chi^{(1)}, \quad \tau_2(\lambda) = \widetilde{W}[\tau_1(\lambda)], \quad \widetilde{W} = \begin{pmatrix} W_{00} & -W_{01} \\ -W_{10} & W_{11} \end{pmatrix}.$$

Proof. Let $\widehat{f} = \widehat{f}_1 \oplus \widehat{f}_2 \in \widetilde{A}$. Then one obtains

$$(6.15) \quad \{\Gamma_0^j \widehat{f}_1, -\Gamma_1^j \widehat{f}_1\} \in \chi^j(\widehat{f}_2) \quad (j = 1, 2).$$

The formula (6.14) is implied by (6.15) and the following equality

$$\begin{pmatrix} \Gamma_0^2 \widehat{f}_1 \\ -\Gamma_1^2 \widehat{f}_1 \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \begin{pmatrix} \Gamma_0^2 \widehat{f}_1 \\ \Gamma_1^2 \widehat{f}_1 \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} W \begin{pmatrix} \Gamma_0^1 \widehat{f}_1 \\ \Gamma_1^1 \widehat{f}_1 \end{pmatrix} = \widetilde{W} \begin{pmatrix} \Gamma_0^1 \widehat{f}_1 \\ -\Gamma_1^1 \widehat{f}_1 \end{pmatrix}.$$

The latter formula from (6.14) is implied by Lemma 3.10. \square

6.3. The double Weyl function. It is shown that associated with every selfadjoint extension \widetilde{A} of A there is a special boundary relation involving the linear relation $A^* \oplus T_2$ and whose parameter space has double dimension. The corresponding Weyl function of the operator $A \oplus S_2$ can be written in the block form and as such is frequently encountered in boundary-eigenvalue problems with boundary conditions depending on the eigenvalue parameter (see e.g. [20], [21]).

Theorem 6.10. *Let A be a symmetric operator in \mathfrak{H}_1 and let $\Pi = \{\mathcal{H}_1, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^* with the Weyl function $M(\lambda)$. Let S_2 be a symmetric operator in a Hilbert space \mathfrak{H}_2 , let $\chi : \mathfrak{H}_2^2 \mapsto \mathcal{H}^2$ be a boundary relation for S_2^* with the domain $\text{dom } \chi = T_2$ and the Weyl family $\tau(\lambda) = \{\phi, \psi\} \in \widetilde{R}(\mathcal{H})$ and let $\widetilde{\mathfrak{H}} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$. Then:*

(i) *the linear relation $\Gamma^{coupl} : \widetilde{\mathfrak{H}}^2 \mapsto \mathcal{H}_\Omega^2$ given by*

$$(6.16) \quad \Gamma^{coupl} = \left\{ \left\{ \widehat{f}_1 \oplus \widehat{f}_2, \begin{pmatrix} h' + \Gamma_1 \widehat{f}_1 \\ h - \Gamma_0 \widehat{f}_1 \end{pmatrix} \oplus \begin{pmatrix} -\Gamma_0 \widehat{f}_1 \\ h' \end{pmatrix} \right\} : \widehat{f}_1 \in A^*, \quad \left\{ \widehat{f}_2, \begin{pmatrix} h \\ h' \end{pmatrix} \right\} \in \chi \right\},$$

is a boundary relation for $A^ \oplus S_2^*$, which satisfies (B1)–(B3) (see Proposition 4.8);*

(ii) *the corresponding Weyl function $M_{coupl}(\cdot)$ belongs to the class $R[\mathcal{H}]$ and is given by*

$$(6.17) \quad M_{coupl}(\lambda) = \begin{pmatrix} -\Phi(\Psi + M\Phi)^{-1} & I - \Phi(\Psi + M\Phi)^{-1}M \\ \Psi(\Psi + M\Phi)^{-1} & \Psi(\Psi + M\Phi)^{-1}M \end{pmatrix}.$$

Proof. (i) Clearly, the linear relation

$$(6.18) \quad \widetilde{\Gamma} = \left\{ \left\{ \widehat{f}_1 \oplus \widehat{f}_2, \left\{ \begin{pmatrix} \Gamma_0 \widehat{f}_1 \\ -h' \end{pmatrix}, \begin{pmatrix} \Gamma_1 \widehat{f}_1 \\ h \end{pmatrix} \right\} \right\} : \quad \widehat{f}_1 \in A^*, \quad \left\{ \widehat{f}_2, \begin{pmatrix} h \\ h' \end{pmatrix} \right\} \in \chi \right\},$$

forms a boundary relation for $A^* \oplus T_2$ and the corresponding Weyl family is

$$\widetilde{\tau}(\lambda) = M(\lambda) \oplus (-\tau(\lambda)^{-1}) = \{I \oplus (-\psi(\lambda)), M(\lambda) \oplus \phi(\lambda)\}.$$

Let W be a $J_{\mathcal{H}_\Omega}$ -unitary operator defined by

$$(6.19) \quad W = \begin{pmatrix} W_{00} & I_{\mathcal{H}^2} \\ -I_{\mathcal{H}^2} & 0 \end{pmatrix}, \quad W_{00} = \begin{pmatrix} 0 & -I_{\mathcal{H}} \\ -I_{\mathcal{H}} & 0 \end{pmatrix}.$$

By Lemma 3.10 $\Gamma^{coupl} = W\widetilde{\Gamma}$ is a new boundary relation Π^{coupl} for $A^* \oplus S_2^*$ whose Weyl family takes the form

$$(6.20) \quad M_{coupl}(\lambda) = W[\widetilde{\tau}(\lambda)] = \left\{ \Omega_0(\lambda), \begin{pmatrix} -I & 0 \\ 0 & \psi(\lambda) \end{pmatrix} \right\}, \quad \text{where} \quad \Omega_0 = \begin{pmatrix} M(\lambda) & \psi(\lambda) \\ -I & \phi(\lambda) \end{pmatrix}.$$

Since $\Omega_0(\lambda)$ is invertible, see [14, Proposition A5], this implies that $\Gamma_0^\Omega \upharpoonright (\widehat{\mathfrak{N}}_\lambda(A) \oplus \widehat{\mathfrak{N}}_\lambda(T))$ is a surjective mapping and by Proposition 4.8 Γ^{coup} is a boundary relation for $A \oplus S_2$ which satisfies (B1)–(B3). This proves the statement (i).

(ii) Setting $\omega := (\psi + M\phi)^{-1}$ one easily derives from (6.20) the formula for the corresponding Weyl function $M_{coup}(\cdot)$:

$$\begin{aligned} M_{coup}(\lambda) &= \begin{pmatrix} -I & 0 \\ 0 & \psi(\lambda) \end{pmatrix} \Omega_0(\lambda)^{-1} \\ &= \begin{pmatrix} -I & 0 \\ 0 & \psi(\lambda) \end{pmatrix} \begin{pmatrix} \phi(\lambda)\omega(\lambda) & \phi(\lambda)\omega(\lambda)M(\lambda) - I \\ \omega(\lambda) & \omega(\lambda)M(\lambda) \end{pmatrix} \\ &= \begin{pmatrix} -\Phi(\Psi + M\Phi)^{-1} & I - \Phi(\Psi + M\Phi)^{-1}M \\ \Psi(\Psi + M\Phi)^{-1} & \Psi(\Psi + M\Phi)^{-1}M \end{pmatrix}. \end{aligned}$$

Moreover, by Proposition 4.8 $M_{coup}(\cdot) \in R[\mathcal{H}]$. This gives (ii). \square

Remark 6.11. (i) If the boundary relation χ in Theorem 6.3 is single-valued then it can be decomposed into a boundary triplet $\Pi'' = \{\mathcal{H}, \chi_0, \chi_1\}$, where the boundary operators χ_j are given by

$$\chi_j = \pi_j \chi : T_2 \rightarrow \mathcal{H}, \quad j = 0, 1.$$

In this case the boundary relation $\widetilde{\Gamma}$ of the form (6.18) becomes a boundary triplet $\widetilde{\Pi} = \{\mathcal{H}^2, \widetilde{\Gamma}_0, \widetilde{\Gamma}_1\}$ where

$$\widetilde{\Gamma} = \begin{pmatrix} \widetilde{\Gamma}_0 \\ \widetilde{\Gamma}_1 \end{pmatrix}, \quad \widetilde{\Gamma}_0 = \begin{pmatrix} \Gamma_0 \\ -\chi_1 \end{pmatrix}, \quad \text{and} \quad \widetilde{\Gamma}_1 = \begin{pmatrix} \Gamma_1 \\ \chi_0 \end{pmatrix},$$

and the equality (6.5) takes the form

$$(6.21) \quad \widetilde{A} = \ker(\widetilde{\Gamma}_1 - B\widetilde{\Gamma}_0) \quad \text{with} \quad B = \begin{pmatrix} 0 & I_{\mathcal{H}} \\ I_{\mathcal{H}} & 0 \end{pmatrix}.$$

In other words the coupling \widetilde{A} is determined by

$$(6.22) \quad \widetilde{A} = \left\{ \widehat{f}_1 \oplus \widehat{f}_2 \in A^* \oplus T_2 : \Gamma_0 \widehat{f}_1 - \chi_0 \widehat{f}_2 = \Gamma_1 \widehat{f}_1 + \chi_1 \widehat{f}_2 = 0 \right\}.$$

In such a form a construction of the coupling \widetilde{A} of two boundary triplets has been introduced in [12] under an additional assumption that $\Pi'' = \{\mathcal{H}, \chi_0, \chi_1\}$ is an ordinary boundary triplet.

(ii) Suppose that in Theorem 6.10 the Nevanlinna family $\tau(\cdot)$ belongs to $R^s(\mathcal{H})$. Then due to (6.21) the Weyl function corresponding to the triplet $\Pi^{coup} = \{\mathcal{H}^2, \widetilde{\Gamma}_1 - B\widetilde{\Gamma}_0, -\widetilde{\Gamma}_0\}$ is $(B - \widetilde{\tau}(\cdot))^{-1}$. Using the Frobenius formula we easily get

$$(6.23) \quad (B - \widetilde{\tau}(\cdot))^{-1} = \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} M & 0 \\ 0 & -\tau^{-1} \end{pmatrix} \right)^{-1} = \begin{pmatrix} -(\tau + M)^{-1} & (\tau + M)^{-1}\tau \\ \tau(\tau + M)^{-1} & (\tau^{-1} + M^{-1})^{-1} \end{pmatrix} = M_{coup}.$$

Note that the matrix of linear fractional transformation $\widetilde{\tau}(\cdot) \rightarrow (B - \widetilde{\tau}(\cdot))^{-1}$ coincides with the block matrix W determined by (6.19), that is $(B - \widetilde{\tau}(\cdot))^{-1} = W[\widetilde{\tau}(\cdot)]$.

Comparing (6.17) with (6.23) we see that in this case M_{coup} coincides with the Weyl function corresponding to the boundary triplet Π^{coup} . Moreover, these reasonings, borrowed from [12], explain the appearance of the linear fractional transformation W in formula (6.20).

Under an additional assumption that $\Pi'' = \{\mathcal{H}, \chi_0, \chi_1\}$ is an ordinary boundary triplet Theorem 6.10 has been proved in our paper [12].

(iii) In the case of finite defect numbers the function $M_{coul}(\cdot)$ appears for instance in the connection of Sturm-Liouville operators and Hamiltonian systems with “ λ -depending” boundary conditions which are expressed by means of a Nevanlinna pair $\{\Phi(\cdot), \Psi(\cdot)\}$ (equivalent to $\tau(\cdot)$). In this case the function $M_{coul}(\cdot)$ is known as the spectral matrix induced by the Nevanlinna pair $\{\Phi(\cdot), \Psi(\cdot)\}$; cf. [20], [21].

Finally, we demonstrate applicability of some results on intermediate extensions from Section 5 when applied to the Weyl function $M_{coul}(\cdot)$ in Theorem 6.10. Namely, we can prove that the diagonal elements of the matrix $M_{coul}(\lambda)$ are also Weyl families of some intermediate extensions of the operator $A \oplus S_2$. In particular, this result gives a geometric interpretation of the Nevanlinna function $(\tau(\cdot) + M(\cdot))^{-1}$ appearing in the Krein-Naimark formula for generalized resolvents (see (7.6)), as a Weyl function of some intermediate extension. The importance of this result is demonstrated in Section 7.2.

Theorem 6.12. *Under the assumptions of Theorem 6.10 the following statements hold:*

(i) *the linear relation*

$$(6.24) \quad H^{(1)} = \left\{ \widehat{f}_1 \oplus \widehat{f}_2 \in S_1^* \oplus T_2 : \Gamma_0 \widehat{f}_1 = 0, \left\{ \widehat{f}_2, \begin{pmatrix} 0 \\ -\Gamma_1 \widehat{f}_1 \end{pmatrix} \right\} \in \chi \right\},$$

is a closed symmetric linear relation in $\widetilde{\mathfrak{H}} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$;

(ii) *the linear relation $\widetilde{\Gamma}^{(1)} : \widetilde{\mathfrak{H}}^2 \mapsto \mathcal{H}^2$ given by*

$$(6.25) \quad \Gamma^{(1)} = \left\{ \widehat{f}_1 \oplus \widehat{f}_2, \begin{pmatrix} \Gamma_1 \widehat{f}_1 + h' \\ -\Gamma_0 \widehat{f}_1 \end{pmatrix} : \left\{ \widehat{f}_2, \begin{pmatrix} \Gamma_0 \widehat{f}_1 \\ h' \end{pmatrix} \right\} \in \chi \right\}.$$

is a boundary relation for $H^{(1)}$ which satisfies the conditions (B1)–(B3);*

(iii) *the Weyl function $M^{(1)}(\lambda)$ of $H^{(1)}$ corresponding to the boundary relation $\widetilde{\Gamma}^{(1)}$ is given by*

$$(6.26) \quad M^{(1)}(\lambda) = -\Phi(\lambda)(\Psi(\lambda) + M(\lambda)\Phi(\lambda))^{-1}.$$

(iv) *the linear relation*

$$(6.27) \quad H^{(2)} = \left\{ \widehat{f}_1 \oplus \widehat{f}_2 \in A^* \oplus T_2 : \Gamma_1 \widehat{f}_1 = 0, \left\{ \widehat{f}_2, \begin{pmatrix} -\Gamma_0 \widehat{f}_1 \\ 0 \end{pmatrix} \right\} \in \chi \right\},$$

is a closed symmetric linear relation in $\widetilde{\mathfrak{H}} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$;

(v) *the linear relation $\widetilde{\Gamma}^{(1)} : \widetilde{\mathfrak{H}}^2 \mapsto \mathcal{H}^2$ given by*

$$(6.28) \quad \Gamma^{(2)} = \left\{ \widehat{f}_1 \oplus \widehat{f}_2, \begin{pmatrix} -\Gamma_0 \widehat{f}_1 + h \\ -\Gamma_1 \widehat{f}_1 \end{pmatrix} : \left\{ \widehat{f}_2, \begin{pmatrix} h \\ -\Gamma_1 \widehat{f}_1 \end{pmatrix} \right\} \in \chi \right\}.$$

is a boundary relation for $H^{(2)}$ which satisfies the conditions (B1)–(B3);*

(vi) *the Weyl function $M^{(2)}(\lambda)$ of $H^{(2)}$ corresponding to the boundary relation $\widetilde{\Gamma}^{(2)}$ is given by*

$$(6.29) \quad M^{(2)}(\lambda) = \Psi(\lambda)(\Psi(\lambda) + M(\lambda)\Phi(\lambda))^{-1}M(\lambda).$$

Proof. Let us apply Proposition 5.1 to the boundary relation Γ^{coup} in Theorem 6.10. Then the boundary conditions in (5.3) take the form

$$\Gamma_1 \widehat{f}_1 + h' = -\Gamma_0 \widehat{f}_1 + h = \Gamma_0 \widehat{f}_1 = 0,$$

or, equivalently,

$$\Gamma_1 \widehat{f}_1 = -h', \quad \Gamma_0 \widehat{f}_1 = h = 0.$$

Then it follows from Proposition 5.1 that the linear relation $H^{(1)}$ is a closed symmetric linear relation in $\widetilde{\mathfrak{H}}$. The equality (6.25) is implied by (5.4). Due to Proposition 5.1 the Weyl function corresponding to the boundary relation $\Gamma^{(1)}$ is the upper left corner of the block matrix $M_{coup}(\lambda)$.

Similarly, the statements (iv)-(vi) are implied by Theorem 6.10 and Corollary 5.2. \square

7. GENERALIZED RESOLVENTS AND ADMISSIBILITY

7.1. Kreĭn's formula for generalized resolvents. Let A be a symmetric operator in a Hilbert space \mathfrak{H} with equal defect numbers. Let \widetilde{A} be a selfadjoint extension of A in a Hilbert space $\widetilde{\mathfrak{H}}$ containing \mathfrak{H} as a closed subspace. The compression $\mathbf{R}_\lambda = P_{\mathfrak{H}}(\widetilde{A} - \lambda)^{-1} \upharpoonright \mathfrak{H}$ of the resolvent of \widetilde{A} to \mathfrak{H} is said to be a generalized resolvent of A .

Using the coupling method, we easily obtain the classical Kreĭn-Naimark formula, parametrizing all generalized resolvents of A by means of maximal dissipative relations (Nevanlinna pairs) $\tau(\lambda)$. Namely, combining Theorem 6.3, Proposition 6.4 and Theorem 3.7 we arrive at the following formula for generalized resolvents (in the Straus form).

Theorem 7.1. *Let A be a symmetric operator in a Hilbert space \mathfrak{H} and $n_+(A) = n_-(A)$. Let \widetilde{A} be a selfadjoint extension of A in a Hilbert space $\widetilde{\mathfrak{H}} \supset \mathfrak{H}$ and let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^* . Then there is a unique Nevanlinna family $\tau(\lambda) \in \widetilde{R}_{\mathcal{H}}$ such that*

$$(7.1) \quad P_{\mathfrak{H}}(\widetilde{A} - \lambda)^{-1} \upharpoonright \mathfrak{H} = (\widetilde{A}_{\tau(\lambda)} - \lambda)^{-1}.$$

Moreover, for any $h \in \mathfrak{H}$, vector $f_1 = P_{\mathfrak{H}}(\widetilde{A} - \lambda)^{-1}h$ is a solution of the “boundary-value problem” with spectral parameter $\tau(\lambda)$ in “boundary condition”

$$(7.2) \quad \begin{cases} f'_1 - \lambda f_1 = h, & \widehat{f}_1 = \{f_1, f'_1\} \in A^*, \\ \{\Gamma_0 \widehat{f}_1, -\Gamma_1 \widehat{f}_1\} \in \tau(\lambda); \end{cases}$$

Conversely, given $\tau(\lambda) \in \widetilde{R}_{\mathcal{H}}$ there is a minimal selfadjoint extension \widetilde{A} of A in a Hilbert space $\widetilde{\mathfrak{H}} \supset \mathfrak{H}$ such that (7.1) holds.

Proof. (i) Let $\lambda \in \rho(\widetilde{A})$ and let $h \in \mathfrak{H}$. Then there is a vector $\widehat{f} = \begin{pmatrix} f \\ f' \end{pmatrix} \in \widetilde{A}$ such that

$$(7.3) \quad f' - \lambda f = h.$$

Projecting of (7.3) to \mathfrak{H}_1 and \mathfrak{H}_2 gives the following equations

$$(7.4) \quad f'_1 - \lambda f_1 = h, \quad f'_2 - \lambda f_2 = 0,$$

where $f_j = P_{\mathfrak{H}_j} f$, $f'_j = P_{\mathfrak{H}_j} f'$, $j = 1, 2$. It follows from (6.5) that $\left\{ \widehat{f}_2, \begin{pmatrix} \Gamma_0 \widehat{f}_1 \\ -\Gamma_1 \widehat{f}_1 \end{pmatrix} \right\} \in \chi$, where $\widehat{f}_j = \begin{pmatrix} f_j \\ f'_j \end{pmatrix}$. Since $\widehat{f}_2 \in \mathfrak{N}_\lambda(T_2)$ this implies

$$(7.5) \quad \{\Gamma_0 \widehat{f}_1, -\Gamma_1 \widehat{f}_1\} \in \tau(\lambda),$$

where $\tau(\cdot)$ is the Weyl family of S_2 corresponding to the boundary relation χ . This proves the statement (i).

(ii) Conversely, starting with $\tau(\cdot)$ and applying Theorem 3.7 we find a simple symmetric operator S_2 in \mathfrak{H}_2 and a minimal boundary relation $\chi : \mathfrak{H}_2^2 \rightarrow \mathcal{H}^2$ for S_2^* such that the corresponding Weyl family is $\tau(\lambda)$. Then by Theorem 6.3 the linear relation \tilde{A} in $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$ (a coupling of T_1 and S_2) defined by (6.5) is an exit space selfadjoint extension of A which satisfies (6.1) and (7.1) with some $\tau_1(\cdot) \in \tilde{R}(\mathcal{H})$ in place of $\tau(\cdot)$. By Proposition 6.4 $\tau_1(\cdot) = \tau(\cdot)$. \square

Combining Theorem 7.1 with formula (4.6) for canonical resolvents we arrive at the following

Theorem 7.2. ([34]). *Let A be a symmetric operator in \mathfrak{H} with $n_+(A) = n_-(A)$, let $\Pi = \{\mathcal{H}_1, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^* , and let $M(\cdot)$ and $\gamma(\cdot)$ be the corresponding Weyl function and the γ -field. Then the formula*

$$(7.6) \quad \mathbf{R}_\lambda = (A_0 - \lambda)^{-1} - \gamma(\lambda)(M(\lambda) + \tau(\lambda))^{-1}\gamma(\bar{\lambda})^*, \quad \lambda \in \rho(A_0) \cap \rho(\tilde{A})$$

with $A_0 = \ker \Gamma_0$ establishes a bijective correspondence between the generalized resolvents \mathbf{R}_λ of A and Nevanlinna families $\tau(\cdot) \in \tilde{R}_{\mathcal{H}}$.

Proof. Let $\lambda \in \rho(A_0)$. According to Proposition 4.2 $\lambda \in \rho(A_{-\tau(\lambda)})$ if and only if $0 \in \rho(M(\lambda) + \tau(\lambda))$. In this case (see (4.6))

$$(7.7) \quad (\tilde{A}_{-\tau(\lambda)} - \lambda)^{-1} = (A_0 - \lambda)^{-1} - \gamma(\lambda)(M(\lambda) + \tau(\lambda))^{-1}\gamma(\bar{\lambda})^*.$$

Now the statement follows from Theorem 7.1. \square

Remark 7.3. (i) Note that for "good" $\tau(\cdot)$ Theorem 7.2 can easily be derived from Theorem 6.10 and formula (4.6) for canonical resolvents with double Weyl function $M_{\text{coupl}}(\cdot)$ (see (6.17)). We explain the proof confining ourself to the case $\tau(\cdot) \in R^u[\mathcal{H}]$. By Theorem 6.12 and Proposition 4.1 there exists an ordinary boundary triplet $\{\mathcal{H}, \chi_0, \chi_1\}$ for S_2^* such that the corresponding Weyl function is $\tau(\cdot)$. Consider a boundary triplet $\{\mathcal{H}^2, \Gamma_0^\Omega, \Gamma_1^\Omega\}$ for $A^* \oplus S_2^*$ of the form (6.16). The corresponding Weyl function $M(\cdot)$ is of the form (6.17), $M(\cdot) = \Omega(\cdot)$. Then \tilde{A} and $A_0 \oplus A_1^2$ ($A_1^2 = \ker \chi_1$) are canonical selfadjoint extensions of $A \oplus S_2$ and the formula (4.6) implies

$$(7.8) \quad (\tilde{A} - \lambda)^{-1} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} (A_0 - \lambda)^{-1} h_1 \\ (A_1^2 - \lambda)^{-1} h_2 \end{pmatrix} - \begin{pmatrix} \gamma(\lambda) & 0 \\ 0 & \gamma^{(2)}(\lambda) \end{pmatrix} \Omega(\lambda) \begin{pmatrix} \gamma(\bar{\lambda})^* h_1 \\ \gamma^{(2)}(\bar{\lambda})^* h_2 \end{pmatrix},$$

where $\gamma^{(2)}(\lambda)$ is the γ -field corresponding to the boundary triplet $\{\mathcal{H}, -\chi_1, \chi_0\}$ and $A_1^{(2)} = \ker \chi_1$. Setting $h_2 = 0$ and applying the projection P_1 onto \mathfrak{H}_1 to (7.8) we arrive at (7.6).

(ii) Note, that in fact, both formulas (7.1) and (7.6) are equivalent to each other and can easily be deduced one from another (cf. [38, 19]).

Remark 7.4. The description of all generalized resolvents was originally given in different forms by M.G. Kreĭn [32] and M.A. Naimark [39]. It has been extended to the case of infinite indices by Saakyan (see [34, 17] and references therein). Another description in a form close to (7.1) was given by A.V. Štraus [46]. A connection of the Kreĭn-Naimark formula with boundary triplets has been discovered in [17], [19], [38]. Moreover, other proofs as well as generalizations of the Kreĭn-Naimark formula for nondensely defined symmetric operators can be found in [19], [38], [22], [37]; see also the references therein.

7.2. Admissibility. In this section some new admissibility criteria will be given, which guarantee that a generalized resolvent of a symmetric operator corresponds to a selfadjoint operator extension. Their relation to some other conditions which have been found earlier in [19], [38], [37] will be discussed.

Let A be a symmetric operator in \mathfrak{H} with equal defect numbers $n_+(A) = n_-(A) < \infty$ and let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^* . According to Theorem 7.2 the generalized resolvents \mathbf{R}_λ of A are in one-to-one correspondence with Nevanlinna families $\tau(\lambda) \in \tilde{R}_{\mathcal{H}}$ via the Kreĭn-Naimark formula (7.6). Let \tilde{A} be a minimal selfadjoint extension of A whose compressed resolvent is equal to \mathbf{R}_λ . Then the family $\tau(\lambda)$ associated to \tilde{A} via (7.6) is said to be Π -admissible, if \tilde{A} is an operator extension of A , i.e., if $\text{mul } \tilde{A} = \{0\}$.

The next theorem gives a general criterion for the Π -admissibility of the family $\tau(\lambda) = \{\phi(\lambda), \psi(\lambda)\}$.

Theorem 7.5. *Let A be a (nondensely defined) closed symmetric operator in \mathfrak{H} with equal defect numbers $n_+(A) = n_-(A) \leq \infty$, let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triple for A^* with Weyl function $M(\lambda)$, and let $\tau(\lambda) = \{\phi(\lambda), \psi(\lambda)\}$ be a Nevanlinna pair in \mathcal{H} . Then:*

(i) *The pair $\{\phi(\lambda), \psi(\lambda)\}$ is Π -admissible if and only if the following two conditions are satisfied:*

$$(7.9) \quad w - \lim_{y \uparrow \infty} \frac{\phi(\lambda)(\psi(iy) + M(iy)\phi(\lambda))^{-1}}{y} = 0$$

and

$$(7.10) \quad \lim_{y \uparrow \infty} \frac{\psi(\lambda)(\psi(iy) + M(iy)\phi(\lambda))^{-1}M(\lambda)}{y} = 0.$$

(ii) *If, in addition, $A_0 = \ker \Gamma_0$ is an operator, then the Π -admissibility of $\{\phi(\lambda), \psi(\lambda)\}$ is equivalent to the single condition (7.9).*
 (iii) *If $A_1 = \ker \Gamma_1$ is an operator, then the Π -admissibility of $\{\phi(\lambda), \psi(\lambda)\}$ is equivalent to the single condition (7.10).*

Proof. (i) By Theorem 3.7 there are a Hilbert space \mathfrak{H}_2 , a symmetric operator S_2 and a boundary relation $\chi : \mathfrak{H}^2 \rightarrow \mathcal{H}^2$ whose Weyl family is $\tau(\lambda) = \{\phi(\lambda), \psi(\lambda)\}$. Let the selfadjoint extension \tilde{A} of $A \oplus S_2$ be as in Lemma 6.10. Moreover, by Lemma 6.10 the function $\Omega(\lambda)$ given by (6.17), is the Weyl function of $A \oplus S_2$ corresponding to the boundary relation $\Gamma_\Omega : \tilde{\mathfrak{H}}^2 \rightarrow \mathcal{H}_\Omega^2$ of the form (6.16). According to Proposition 4.9 the multivalued part of the linear relation \tilde{A} is trivial if and only if

$$(7.11) \quad w - \lim_{y \uparrow \infty} \frac{\Omega(iy)}{y} = 0.$$

Now it remains to note that (7.11) is equivalent to the conditions (7.9), (7.10).

(ii) Assume that A_0 is an operator and consider the boundary relation

$$\Gamma^{(1)} = \left\{ \widehat{f}_1 \oplus \widehat{f}_2, \begin{pmatrix} \Gamma_1 \widehat{f}_1 + h' \\ -\Gamma_0 \widehat{f}_1 \end{pmatrix} : \left\{ \widehat{f}_2, \begin{pmatrix} \Gamma_0 \widehat{f}_1 \\ h' \end{pmatrix} \right\} \in \chi \right\}.$$

for $(H^{(1)})^*$, where $H^{(1)}$ is a closed symmetric linear relation in $\widetilde{\mathfrak{H}} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$ given by (6.24). As was shown in Theorem 6.12 the boundary relation $\widetilde{\Gamma}^{(1)} : \widetilde{\mathfrak{H}}^2 \mapsto \mathcal{H}^2$ satisfies the conditions (B1)-(B3) and the corresponding Weyl function of $H^{(1)}$ is $-(M(\lambda) + \tau(\lambda))^{-1}$. To see that $H^{(1)}$ is an operator assume that $\widehat{f}_1 \oplus \widehat{f}_2 = \{0, f'_1\} \oplus \{0, f'_2\} \in H^{(1)}$. Then $\widehat{f}_1 = 0$, since $\Gamma_0 \widehat{f}_1 = 0$ and $\text{mul } A_0 = \{0\}$. Using the last condition in the definition of $H^{(1)}$ in (6.24) one obtains $\{\widehat{f}_2, 0\} \in \chi$, which due to Proposition 4.8 implies $\widehat{f}_2 \in S_2$. Since S_2 is an operator, it follows that $\widehat{f}_2 = 0$, and hence H is also an operator. In view of Proposition 4.9 \widetilde{A} is an operator if and only if (7.9) holds.

(iii) In the case where $A_1 = \ker \Gamma_1$ is an operator one can replace the boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ by $\widetilde{\Pi} = \{\mathcal{H}, \Gamma_1, -\Gamma_0\}$. Then the corresponding Weyl families are transformed to $\widetilde{M}(\lambda) = -M(\lambda)^{-1}$ and $\widetilde{\tau}(\lambda) = -\tau(\lambda)^{-1}$, and the statement in the part (iii) is obtained from the part (ii). \square

Remark 7.6. Other approaches to the admissibility problem have been proposed in [37], [38] and [19]. Namely, a direct deduction of Theorem 7.5 (ii) from Krein-Naimark formula has been obtained in [38]. This proof is more complicated than the one proposed here. Furthermore, under the additional assumption that $A_1 = \ker \Gamma_1$ is an operator, another criterion of admissibility (with rather complicated proof) has been obtained in [19]. This criterion is equivalent to the statement Theorem 7.5(iii), while we don't know a direct proof of their equivalence.

Another criterion of admissibility (without additional assumptions) has been obtained in [37]. A connection of Theorem 7.5 with the Langer-Textorius result is discussed in Section 7.3.

In the next proposition another admissibility criterion is obtained, when \widetilde{A} is viewed as an extension of the symmetric intermediate extension H_T defined in Proposition 5.4.

Proposition 7.7. *Let A be a simple symmetric operator satisfying the assumptions of Theorem 7.5. Assume that $T \in [\mathcal{H}]$ and let $M_T(\lambda)$ be defined by*

$$(7.12) \quad \begin{aligned} M_T(\lambda) = & -T^*(M(\lambda) + \tau(\lambda))^{-1}T - T^*(M(\lambda) + \tau(\lambda))^{-1}\tau(\lambda) \\ & - \tau(\lambda)(M(\lambda) + \tau(\lambda))^{-1}T + \tau(\lambda)(M(\lambda) + \tau(\lambda))^{-1}M(\lambda). \end{aligned}$$

Then for the extension $A^{(\tau)}$ ($= \widetilde{A}$) in (6.22) to be an operator it is necessary and, if $\widetilde{A}_{T^} = \ker(\Gamma_1 - T^*\Gamma_0)$ is an operator, it is also sufficient that the following condition holds:*

$$(7.13) \quad s - \lim_{y \uparrow \infty} \frac{M_T(iy)}{y} = 0;$$

Proof. It follows from Proposition 5.4 and Theorem 6.10 that $M_T(\lambda)$ is the Weyl function of the linear relation

$$(7.14) \quad H_T = \left\{ \widehat{f}_1 \oplus \widehat{f}_2 \in A^* \oplus S_2^* : \Gamma_1 \widehat{f}_1 + h' = \Gamma_0 \widehat{f}_1 - h = \Gamma_1 \widehat{f}_1 - T^* \Gamma_0 \widehat{f}_1 = 0, \left\{ \widehat{f}_1, \begin{pmatrix} h \\ h' \end{pmatrix} \right\} \in \Delta \right\}$$

corresponding to the boundary relation

$$(7.15) \quad \Gamma^T = \left\{ \left\{ \widehat{f}_1 \oplus \widehat{f}_2, \begin{pmatrix} -\Gamma_0 \widehat{f}_1 + h \\ -T^* \Gamma_0 \widehat{f}_1 + h' \end{pmatrix} \right\} : \widehat{f}_1 \in A^*, \quad \left\{ \widehat{f}_2, \begin{pmatrix} h \\ h' \end{pmatrix} \right\} \in \Delta \right\},$$

The necessity of the condition (7.13) follows immediately from (7.9) and (7.10) in Theorem 7.5. To prove the sufficiency let us show that the following implication holds:

$$(7.16) \quad \text{mul } \widetilde{A}_{T^*} = \{0\} \Rightarrow \text{mul } H_T = \{0\}.$$

Indeed, if $\widehat{f} = (\widehat{f}_1, \widehat{f}_2)^\top \in H_T$ and $\widehat{f}_i = \{0, f'_i\}$, $i = 1, 2$, then (7.14) implies that $\Gamma_1 \widehat{f}_1 - T^* \Gamma_0 \widehat{f}_1 = 0$ and, hence, $\widehat{f}_1 \in \widetilde{A}_{T^*}$. Since $\text{mul } \widetilde{A}_{T^*} = \{0\}$, one obtains $\widehat{f}_1 = 0$. Now it follows from (7.14) that $\{\widehat{f}_2, 0\} \in \Delta$. Thus, $\widehat{f}_2 \in S_2 = \ker \Delta$, and consequently $\widehat{f}_2 = 0$, since S_2 is a simple operator. This proves that, $\text{mul } H_T = \{0\}$. Since $A^{(\tau)} = \ker \Gamma_0^T$, it follows from (7.14) that $M_T(\lambda)$ is a Weyl function of the pair $(H_T, A^{(\tau)})$. By Proposition 4.9 the condition (7.13) implies that $A^{(\tau)}$ is an operator. This completes the proof. \square

7.3. The Langer-Textorius criterion. In this subsection a new proof for the admissibility criterion in [37] will be given.

Following [37] introduce the operator function Q_τ with values in $[\mathcal{H}]$ by

$$(7.17) \quad Q_{LT}^\tau(\lambda; z_0) := M(\lambda) - (M(\lambda) - M(z_0)^*)(M(\lambda) + \tau(\lambda))^{-1}(M(\lambda) - M(z_0)).$$

The function $Q_{LT}^\tau(\lambda; z_0)$ is a Q -function of a pair $(H_{LT}, A^{(\tau)})$, where $A^{(\tau)}$ is a minimal selfadjoint exit space extension of A corresponding to $\tau(\lambda)$ in (7.6) and H_{LT} is a symmetric restriction of $A^{(\tau)}$, cf. [35], [37]. In the following proposition the symmetric linear relation H_{LT} is calculated explicitly. This allows to derive the Langer-Textorius criterion from Proposition 7.7.

The next theorem specifies the operator H_{LT} with the help of boundary operators.

Proposition 7.8. *Let the assumptions be as in Proposition 7.7 and let $z_0 \in \mathbb{C}_+$ be fixed. Then:*

(i) *the linear relation H_{LT} defined by*

$$(7.18) \quad H_{LT} = \left\{ \widehat{f} = \widehat{f}_1 \oplus \widehat{f}_2 \in A^* \oplus S_2^* : \begin{array}{l} \Gamma_1 \widehat{f}_1 + h' = \Gamma_0 \widehat{f}_1 - h = 0, \\ \Gamma_1 \widehat{f}_1 - M(z_0)^* \Gamma_0 \widehat{f}_1 = 0, \end{array} \quad \left\{ \widehat{f}_1, \begin{pmatrix} h \\ h' \end{pmatrix} \right\} \in \Delta \right\}$$

is a closed symmetric operator;

(ii) *a linear relation*

$$(7.19) \quad \Gamma^{LT} = \left\{ \left\{ \widehat{f}_1 \oplus \widehat{f}_2, \begin{pmatrix} -\Gamma_0 \widehat{f}_1 + h \\ -M(z_0)^* \Gamma_0 \widehat{f}_1 + h' \end{pmatrix} \right\} : \widehat{f}_1 \in A^*, \quad \left\{ \widehat{f}_2, \begin{pmatrix} h \\ h' \end{pmatrix} \right\} \in \Delta \right\},$$

is a boundary relation for H_{LT}^ ;*

(iii) *the Weyl function corresponding to Γ^{LT} is given by*

$$(7.20) \quad M_{LT}(\lambda) = Q_{LT}^\tau(\lambda; z_0) - 2\text{Re } M(z_0);$$

(iv) *$\tau(\lambda)$ is admissible if and only if*

$$(7.21) \quad s - \lim_{y \uparrow \infty} \frac{M_{LT}(iy)}{y} = 0.$$

Proof. As was shown in Proposition 7.7 H_{LT} is a closed symmetric linear relation in $\mathfrak{H}_1 \oplus \mathfrak{H}_2$. Moreover, the linear relation $\tilde{A}_{M(z_0)^*}$ takes the form

$$\tilde{A}_{M(z_0)^*} = A + \widehat{\mathfrak{N}}_{\bar{z}_0}.$$

Since $\mathfrak{N}_{\bar{z}_0} \cap \text{dom } A = \{0\}$, this implies that $\tilde{A}_{M(z_0)^*}$ is an operator. Now it follows from (7.16) that H_{LT} is an operator. The expressions (7.18), and (7.19) are obtained from Proposition 5.4 and the identities in (6.16). \square

The functions in (7.17) and (7.20) are related by

$$Q_{LT}^\tau(\lambda; z_0) = M_{LT}(\lambda) + 2\text{Re } M(z_0).$$

Therefore, Proposition 7.8 (iv) yields the following theorem in [37].

Theorem 7.9. ([37]) *Let $z_0 \in \mathbb{C}_+$. Then the minimal selfadjoint extension $A^{(\tau)}$ of A in Krein's formula (7.6) is an operator if and only if*

$$(7.22) \quad \lim_{y \uparrow \infty} \frac{(Q_{LT}^\tau(iy; z_0)h, h)}{y} = 0, \quad h \in \mathcal{H}.$$

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